Received: xxx **Accepted:** xxx **Published:** xxx.
DOI. xxxxxxx
Volume xxx, Issue xxx, p.p. 1-20: xxx

Research Article

Open Access

Control and Optimization in Applied Mathematics - COAM

Scenario Approach-based Parametric Optimal Control for Uncertain Dynamical Systems via the Variational Iteration Method

Ghania Idiri[1]

Laboratoire de Conception et Conduite des Systèmes de Production, Université Mouloud Mammeri, 15 000 Tizi-Ouzou, Algérie

Correspondence:
Ghania Idiri
E-mail:
ghania.idiri@ummto.dz

How to Cite
Idiri, Gh. (2027). "Scenario approach-based parametric optimal control for uncertain dynamical systems via the variational iteration method". *Control and Optimization in Applied Mathematics*, X(x), 1-20.
https://doi.org/10.30473/coam.2026.76729.1376

Abstract. In this paper, a systematic design approach for the parametric optimization of uncertain dynamical systems with bounded parameter uncertainties is proposed. The methodology proceeds in two stages. In the first stage, the control input is expressed as a finite linear combination of polynomial basis functions, and an approximate analytical solution of the state equation is derived using the Variational Iteration Method, which is a semi-analytical iterative technique that avoids discretization and linearization. Substituting this approximate trajectory into the performance index yields a robust min–max optimization problem parameterized by the control coefficients. In the second stage, the robust optimization problem is converted into a tractable scenario optimization problem by drawing a finite number of independent and identically distributed samples from the uncertainty set. The resulting problem is solved using a genetic algorithm. The effectiveness of the proposed approach is demonstrated through two application examples. The first example concerns an uncertain linear-quadratic regulator, and the second addresses an uncertain nonlinear optimal control problem. Optimal control and state trajectories are provided for a set of samples. In addition, the optimal value of the performance index is reported, showing that this value does not exceed the threshold imposed by the proposed approach. The paper concludes by discussing limitations, including dependence on the accuracy of the Variational Iteration Method approximation and the assumption of a known probability distribution over the uncertainty set, and identifies directions for future research.

Keywords. Optimal control, Parametric optimization, Uncertain system, Variational iteration method, Scenario approach.

MSC. 09J15; 49M25; 93B50.

1 Introduction

Optimal control, or dynamic optimization, consists in designing a controller that steers a dynamical system from a given initial state to a final one while optimizing a specified performance index [11, 26, 32, 34]. Optimal control plays a central role in modern control system design and finds widespread application across numerous engineering disciplines [4, 10, 17, 27, 39, 42].

The design of an optimal control for a dynamical system can be achieved using either direct or indirect methods [32]. The principle of direct methods consists of converting the optimal control problem into a nonlinear optimization problem, which is often solved numerically using efficient optimization algorithms [4, 10]. The control parameterization method, the control vector iteration method, and the finite element collocation method are representative examples of this class [11]. Indirect methods, developed based on the theory of the calculus of variations, allow the optimality conditions to be formulated as a set of ordinary differential equations, which can be solved analytically for simple problems or numerically for complex ones [32, 39]. The Euler-Lagrange equation, the minimum principle, and the Hamilton-Jacobi-Bellman equation are well-known and widely used indirect methods [32]. Direct and indirect methods have been successfully applied to find the optimal control of dynamical systems without uncertainties [39].

Optimal control in the presence of uncertainty is a critical issue and represents a promising research field that remains largely unexplored using either direct or indirect methods. To address infinite-horizon optimal control for uncertain quantum-mechanical dynamical systems, [40] introduced a trajectory-informed machine learning approach. In [44], the prescribed-time control approach is extended to the design of optimal control for uncertain nonlinear systems. Robust model predictive control is employed in [2] to address the optimal control problem for dynamical systems under interval uncertainty. Within the framework of reinforcement relearning, [33] proposed a design approach for approximate optimal control of uncertain nonlinear systems.

The parametric optimization approach has been successfully applied to solve optimal control problems for dynamical systems [11, 14, 34, 36, 37]. This approach has been successfully applied to solve optimal control problems for switched systems [23], time-delay systems [12], integer-order systems [13, 14, 21, 24], and fractional-order systems [20]. A survey of the application of the parametric optimization approach to solve optimal control problems can be found in [25, 30].

To the best of our knowledge, the parametric optimization approach has not yet been applied to systems with uncertainties, which constitutes the primary motivation for the present work. In this study, the parametric optimization method is extended to dynamical systems with parameter uncertainties. The parametric optimization method involves finding an optimal control expressed as an expansion in terms of basis functions, after which the optimal control parameters are determined by solving an optimization problem.

The principle of the proposed parametric optimization approach consists of using the Variational Iteration Method (VIM) [18] to convert the optimal control problem into a robust optimization problem. The VIM is a well-established technique used to iteratively solve different types of differential equations through a correction functional [19]. This iterative method has been successfully applied to solve optimal control [1, 3] and predictive control problems [38] using indirect approaches such as the Euler-Lagrange equation and the minimum principle.

The main idea of the present work consists of solving the resulting state equation, obtained by substituting the control expansion into the model of the dynamical system, using the Variational Iteration Method. The obtained approximate solution is then substituted into the performance index to formulate a robust optimization problem. To solve this robust optimization problem, the scenario approach [6, 7, 31] is applied to obtain a scenario optimization problem by extracting independent and identically distributed samples (scenarios) from the uncertainty set according to a known probability distribution [9]. The resulting scenario optimization problem is then solved using efficient optimization methods [29] to determine the optimal control parameters, i.e., the expansion coefficients. Application examples are provided to illustrate the proposed design approach and demonstrate its effectiveness. The proposed method is not intended to replace optimal control approaches, but rather to extend parametric optimization techniques to uncertain dynamical systems, where analytical or approximate optimal solutions may not be directly available or easily derivable.

The proposed design approach proceeds in two main steps. The first step consists of converting the optimal control problem into a robust optimization problem. This can be readily achieved even when the dynamical system to be controlled is nonlinear, since an approximate solution of the state equation can be obtained using the Variational Iteration Method without discretization or linearization [18, 19]. In the second step, the robust optimization problem is reduced to a scenario optimization problem using the scenario approach [7]. This approach provides a probabilistic approximation of the robust optimization problem and has been successfully applied to solve several control problems [5, 6]. These considerations motivate the use of the Variational Iteration Method and the scenario approach in developing the proposed design methodology for parametric optimization of uncertain dynamical systems.

The rest of the paper is organized as follows. In Section 2, we formulate the optimal control problem for an uncertain dynamical system. Section 3 presents the two mathematical tools used in this work, namely the Variational Iteration Method and the scenario approach. In Section 4, we describe and summarize the steps of the proposed design approach. Two application examples are provided in Section 5, and Section 6 concludes the paper.

2 Control problem statement

Consider an uncertain dynamical system

$$\dot{x}(t) = f(x(t),\, u(t),\, t,\, \theta), \tag{1}$$

with the initial condition

$$x(t_0) = x_0, \tag{2}$$

where $t \in [t_0,\, t_f]$ is the time variable, with t_0 and t_f denoting the initial and final times, respectively. The interval $[t_0,\, t_f]$ defines the control horizon. The state vector is $x \in \mathbb{R}^n$ ($\mathbb{R}$ being the set of real numbers), and the control variable is $u \in \mathbb{R}$. θ is the vector of uncertain parameters ranging within the bounded set $\Delta = [\theta_{\min},\, \theta_{\max}]$. The positive integers n and p represent the dimensions of the vectors x and θ, respectively. The function f is a continuously differentiable vector-valued function.

The objective is to design an optimal control law $u(t)$ that optimizes the following performance index:

$$J(u(t)) = \psi(x(t_f),\, t_f) + \int_{t_0}^{t_f} \phi(x(t),\, u(t),\, t)\, dt, \tag{3}$$

where ψ and ϕ are continuously differentiable scalar functions.

The optimal control problem can be stated as the following dynamic optimization problem.

$$\min_{u(t)} J(u(t)) = \psi(x(t_f),\, t_f) + \int_{t_0}^{t_f} \phi(x(t),\, u(t),\, t)\, dt \tag{4}$$

subject to:

$$\dot{x}(t) = f(x(t),\, u(t),\, t,\, \theta), \tag{5}$$

$$x(t_0) = x_0, \tag{6}$$

$$\theta_{\min} \le \theta \le \theta_{\max}. \tag{7}$$

3 Mathematical tools

In this section, we briefly introduce the Variational Iteration Method [18, 19] and the scenario optimization technique [7]. The former is used to solve differential equations, while the latter is employed to address robust optimization problems. These two mathematical tools are applied to reduce the robust optimization problem (4)–(7) to a scenario optimization problem.

3.1 Variational iteration method

The variational iteration method is a semi-analytical method that has been successfully used to find solutions to various types of differential and integro-differential equations [8, 18, 19, 28]. Using this iterative method, one can obtain successive approximations that converge to the exact solution. In this work, we deal with dynamical systems described by first-order ordinary differential equations. Hence, to illustrate the Variational Iteration Method, we consider a first-order ordinary differential equation

$$\dot{x}(t) = F(x(t),\, t), \tag{8}$$

where F is a Lipschitz nonlinear function. Equation (8) can be written in the following operator form

$$\mathcal{L}\, x(t) + \mathcal{N}(x(t)) = g(t), \tag{9}$$

where $\mathcal{L}$ is a linear differential operator, $\mathcal{N}$ is the nonlinear operator, and g is a continuous function.

To find successive approximations that converge to the solution $x(t)$, we use the following correction functional [18, 19]:

$$x^{(j)}(t) = x^{(j-1)}(t) + \int_0^t \lambda(s) \left(\mathcal{L}\, x^{(j-1)}(s) + \mathcal{N}\left(\tilde{x}^{(j-1)}(s)\right) - g(s)\right) ds, \tag{10}$$

where λ is the Lagrange multiplier to be identified using the variational theory [3, 18], and $\tilde{x}^{(j)}(t)$ is the restricted variation, that is, the variation $\delta\tilde{x}^{(j)}(t) = 0$. The initial approximation $x^{(0)}(t)$ is chosen such that the initial condition (6) is satisfied. Hence, when an exact solution exists, it can be obtained as follows

$$x(t) = \lim_{j\to\infty} x^{(j)}(t). \tag{11}$$

On the other hand, an approximate solution can be achieved after a few iterations. The accuracy of the approximation can be evaluated as follows

$$\|x^{(j)}(t) - x^{(j-1)}(t)\|_{L^2([t_0,\, t_f])} = \int_0^t \left(x^{(j)}(t) - x^{(j-1)}(t)\right)^2 dt \leq \varepsilon, \tag{12}$$

where ε is a given threshold and L^2 is the Lebesgue space of square-integrable functions [43]. In this case, the approximation of the solution is given as follows

$$x(t) \approx x^{(j)}(t). \tag{13}$$

Remark 1. The correction functional (10) can be written under the following form

$$x^{(j)}(t) = A\,x^{(j-1)}(t), \tag{14}$$

with the operator A defined as follows

$$A\,x(t) = x(t) + \int_0^t \mathcal{L}\,x(s) + \mathcal{N}(\tilde{x}(s)) - g(s)\,ds. \tag{15}$$

Then, according to [35, Theorem 2.3], if the operator A is contractive, the resulting successive approximations $x^{(j)}(t)$ converge to the exact solution $x(t)$. In the following, it is assumed that this condition on A is fulfilled.

3.2 Scenario approach

The scenario approach is a probabilistic approximation for robust optimization problems [5, 6, 16, 22, 41]. In this section, we briefly introduce the principle of this approach; more details can be found in [7].

Let us consider the following robust optimization problem

$$\min_{z\in\Re^n}\max_{\theta\in\Delta} \mathcal{F}(z,\,\theta), \tag{16}$$

where $z \in \Re^n$ is the vector of decision (optimization) variables, and θ denotes the bounded set of uncertain parameters that affect the objective function $\mathcal{F}$. The integer n denotes the dimension of the decision vector space.

Now, let us assume that $\mathcal{F}(z,\,\theta)$ is convex with respect to z, hence we have

$$\mathcal{F}(z,\,\theta) \leq y, \quad \forall z \in \Re^n, \tag{17}$$

consequently, the robust optimization problem (16) can be rewritten in the following form [6]

$$\min_{z,\,y} y \tag{18}$$

subject to:

$$\mathcal{F}(z,\,\theta) \leq y, \tag{19}$$

or equivalently

$$\min_{w} y \tag{20}$$

subject to:

$$\mathcal{G}(w,\,\theta) \leq 0, \tag{21}$$

where $w \in \Re^d$ $(d = n+1)$ is the new vector of decision variables, defined as follows:

$$w = \begin{bmatrix} z \\ y \end{bmatrix}. \tag{22}$$

To find the solution of the resulting robust optimization problem (20–(21), we assume that N independent and identically distributed samples θ_i $(i = 1, \ldots, N)$ can be extracted from θ according to a probability distribution $\mathbb{P}$ [5, 6]. Hence, the robust optimization problem (20)–(21) can be approximated or relaxed as follows [6]:

$$\min_w y \tag{23}$$

subject to:

$$\mathcal{G}(w, \theta_i) \leq 0, \quad i = 1, \ldots, N. \tag{24}$$

Optimization algorithms [29] can then be employed to find the solution w^* of the scenario optimization problem (23)–(24).

Note that the number N of scenarios or samples required to achieve the solution is given by the following theorem.

Theorem 1. [6] For any $\epsilon \in (0, 1)$ (risk parameter) and $\beta \in (0, 1)$ (confidence parameter), if the number of scenarios N satisfies

$$N \geq \frac{2}{\epsilon}\left(\ln\left(\frac{1}{\beta}\right) + d\right). \tag{25}$$

Then, with probability at least $1 - \beta$, it holds that w^* is ϵ-risk guaranteed, that is,

$$\mathbb{P}\left(\theta \in \Delta : \mathcal{G}(w^*, \theta) > 0\right) \leq \epsilon. \tag{26}$$

4 Proposed design approach

To find the optimal control $u(t)$, we use the parametric optimization technique. The principle of this technique consists of approximating the control variable $u(t)$ as follows [11, 14, 20, 21, 34, 36].

$$u(t) = \sum_{i=1}^{q} k_i\, \vartheta_i(t), \tag{27}$$

where $\vartheta_i(t)$ are well-known appropriate basis functions [34] and k_i $(i = 1, \ldots, q)$ are the control parameters. Thus, the optimization is carried out with respect to control parameters k_i which are to be determined. In the following, a design approach is proposed to determine the optimal control parameters k_i ($i = 1, \ldots, q$) of the control law (27). The main idea is to convert the uncertain optimal control problem (4)–(7) into a robust optimization problem using the variational iteration method, which is then approximated via the scenario approach.

By substituting the control law (27) into the state equation (5), the following ordinary differential equation is obtained:

$$\dot{x}(t) = f\left(x(t), \sum_{i=1}^{q} k_i\, \vartheta_i(t),\, t,\, \theta\right), \tag{28}$$

which reduces to the following form;

$$\dot{x}(t) = h\left(x(t),\, t,\, k,\, \theta\right), \tag{29}$$

where k is the vector of control parameters k_i ($i = 1, \ldots, q$), that is,

$$k = [k_1,\, k_2,\, \ldots,\, k_q]\,. \tag{30}$$

Now, Equation (28) can be written in the following operator form

$$\mathcal{L}\, x(t) + \mathcal{N}(x(t)) = g(t,\, k,\, \theta), \tag{31}$$

where $\mathcal{L}$ and $\mathcal{N}$ are the linear differential operator and nonlinear operator, respectively. Note that these operators may explicitly depend on both the control parameters k and the system parameters θ.

Then, by applying the variational iteration method, an accurate approximation of the solution of the ordinary differential equation (29) can be obtained using the following correction functional

$$x^{(j)}(t) = x^{(j-1)}(t) + \int_0^t \lambda(s) \left(\mathcal{L}\, x^{(j-1)}(s) + \mathcal{N}(\tilde{x}^{(j-1)}(s)) - g(s,\, k,\, \theta)\right)\, ds, \tag{32}$$

which yields an approximation of the solution $x(t)$ as a function of the variables t, k, and θ, that is,

$$x(t) \approx x^{(j^*)}(t) = \Phi\left(t,\, k,\, \theta\right), \tag{33}$$

where j^* is the number of iterations required to achieve convergence and to obtain an accurate approximation of the performance index (4).

Remark 2. Since the resulting ordinary differential equation (29) depends on the unknown control parameters k_i, the convergence analysis of the VIM method becomes non-trivial, as it prevents a direct verification of condition (12). Nevertheless, a suitable value of j^* can be determined by verifying condition (12) for representative values of k_i, and this value is then retained for the parametric case.

Now, substituting both the solution (33) and the control law (27) into the performance index (4) yields

$$\tilde{J}\,(k,\,\theta) = \psi\left(x^{(j^*)}(t_f),\,t_f\right) + \int_{t_0}^{t_f} \phi\left(x^{(j^*)}(t),\,\sum_{i=1}^{q} k_i\,\vartheta_i(t),\,t\right)\,dt. \tag{34}$$

Hence, the optimal control problem (4)–(7) reduces to the following robust optimization problem.

$$\min_{k}\max_{\theta}\tilde{J}\,(k,\,\theta) \tag{35}$$

subject to:

$$\theta_{\text{min}} \le \theta \le \theta_{\text{max}}. \tag{36}$$

Then, by extracting N independent and identically distributed samples θ_i $(i = 1, \ldots, N)$ from θ, the following scenario optimization problem is obtained.

$$\min_{k,\,y} y \tag{37}$$

subject to:

$$\tilde{J}\,(k,\,\theta_i) - y \le 0, \quad i = 1, \ldots, N, \tag{38}$$

or equivalently

$$\min_{w} y \tag{39}$$

subject to:

$$\mathcal{G}\,(w,\,\theta_i) \le 0, \quad i = 1, \ldots, N, \tag{40}$$

with

$$w = \begin{bmatrix} k \\ y \end{bmatrix}. \tag{41}$$

The solution of the scenario optimization problem, obtained using optimization methods [29], yields the optimal values of the parameters k, that is, k^*. The robust optimal control is then given as follows:

$$u^*(t) = \sum_{i=1}^{q} k_i^*\,\vartheta_i(t). \tag{42}$$

Thus, the various steps of the proposed design approach can be summarized as follows:

Step 1. Approximate the control variable as in (27) by selecting the basis functions $\vartheta_i(t)$ and the positive integer q,

Step 2. Substitute the control approximation (27) into the state equation (5).

Step 3. Solve the resulting ordinary differential equation, using the variational iteration method, to obtain an approximation of the solution,

Step 4. Substitute the solution obtained in Step 3 into the performance index to obtain the robust optimization problem (35)–(36),

Step 5. Reduce the robust optimization problem obtained in Step 4 to the scenario optimization problem (39)–(40) by assuming N independent and identically distributed samples θ_i ($i = 1, \ldots, N$) extracted from θ,

Step 6. Solve the scenario optimization problem (39)–(40) using an optimization method to obtain the optimal parameters k_i^* of the optimal control.

5 Application examples

In this section, two application examples are presented. The first example is described in detail to illustrate the various steps of the proposed design approach. The samples are generated by assuming that the uncertainties are normally distributed. The scenario optimization problem is solved using genetic algorithms [15, 29].

Note that, since the time interval $[0, t_f]$ is compact, according to the Weierstrass approximation theorem [43], the control function $u(t)$ can be accurately approximated by a polynomial function. Therefore, in the following, the basis functions considered are of the form

$$\vartheta_j(t) = t^{j-1}. \tag{43}$$

Remark 3. For large q, monomial basis functions in (43) become numerically ill-conditioned. Alternatively, Chebyshev and Legendre polynomials can be used.

Example 1.

Consider the following uncertain linear-quadratic optimal control problem

$$\min_{u(t)} J(u(t)) = \int_0^2 x^2(t) + u^2(t)\, dt \tag{44}$$

subject to:

$$\dot{x}(t) = \theta\, x(t) + u(t), \tag{45}$$

$$x(0) = 0, \tag{46}$$

$$-3 \leq \theta \leq -1. \tag{47}$$

We seek a control law of the form

$$u(t) = \sum_{i=1}^{3} k_i\, t^{i-1} = k_1 + k_2\, t + k_3\, t^2. \tag{48}$$

Hence, substituting the control (48) into the system to be controlled (45),

$$\dot{x}(t) = \theta\, x(t) + \left(k_1 + k_2\, t + k_3\, t^2\right), \tag{49}$$

which can be solved iteratively using the following correction functional:

$$x^{(j+1)}(t) = x^{(j)}(t) + \int_0^t \lambda(s) \left(\dot{x}^{(j)}(s) - \theta\, x^{(j)}(s) - \left(k_1 + k_2\, s + k_3\, s^2\right)\right)\, ds, \tag{50}$$

The identification of the Lagrange multiplier using the calculus of variations [3, 18] yields $\lambda(t) = -1$; hence,

$$x^{(j+1)}(t) = x^{(j)}(t) - \int_0^t \left(\dot{x}^{(j)}(s) - \theta\, x^{(j)}(s) - \left(k_1 + k_2\, s + k_3\, s^2\right)\right)\, ds. \tag{51}$$

This, after performing two iterations, yields the approximate solution:

$$x(t) \approx x^{(2)}(t) = k_1\, t + \frac{k_2\, t^2}{2} + \frac{k_3\, t^3}{3} + \frac{\theta\, t^2\, \left(k_3\, t^2 + 2\, k_2\, t + 6\, k_1\right)}{12}. \tag{52}$$

By substituting the control (48) and the approximate solution (52) into the performance index (44), we obtain the following robust optimization problem:

$$\min_k \max_\theta \tilde{J}(k,\, \theta) \tag{53}$$

subject to:

$$-3 \leq \theta \leq -1, \tag{54}$$

where $k = [k_1\ k_2\ k_3]^T$ and

$$\begin{aligned}\tilde{J}(k,\, \theta) =& \frac{8\, \theta^2\, {k_1}^2}{5} + \frac{16\, \theta^2\, k_1\, k_2}{9} + \frac{32\, \theta^2\, k_1\, k_3}{21} + \frac{32\, \theta^2\, {k_2}^2}{63} + \frac{8\, \theta^2\, k_2\, k_3}{9} \\ &+ \frac{32\, \theta^2\, {k_3}^2}{81} + 4\, \theta\, {k_1}^2 + \frac{16\, \theta\, k_1\, k_2}{3} + \frac{16\, \theta\, k_1\, k_3}{3} + \frac{16\, \theta\, {k_2}^2}{9} \\ &+ \frac{32\, \theta\, k_2\, k_3}{9} + \frac{16\, \theta\, {k_3}^2}{9} + \frac{14\, {k_1}^2}{3} + 8\, k_1\, k_2 + \frac{48\, k_1\, k_3}{5} + \frac{64\, {k_2}^2}{15} \\ &+ \frac{104\, k_2\, k_3}{9} + \frac{2656\, {k_3}^2}{315}.\end{aligned} \tag{55}$$

Now, by assuming N independent and identically distributed samples θ_i $(i = 1, \dots, N)$, extracted from θ, the robust optimization problem (53)–(54) is approximated by the scenario optimization problem

$$\min_{w} y \tag{56}$$

subject to:

$$\begin{aligned}
&\frac{8\,\theta_i^2\,{k_1}^2}{5} + \frac{16\,\theta_i^2\,k_1\,k_2}{9} + \frac{32\,\theta_i^2\,k_1\,k_3}{21} + \frac{32\,\theta_i^2\,{k_2}^2}{63} + \frac{8\,\theta_i^2\,k_2\,k_3}{9} \\
&+ \frac{32\,\theta_i^2\,{k_3}^2}{81} + 4\,\theta_i\,{k_1}^2 + \frac{16\,\theta_i\,k_1\,k_2}{3} + \frac{16\,\theta_i\,k_1\,k_3}{3} + \frac{16\,\theta_i\,{k_2}^2}{9} \\
&+ \frac{32\,\theta_i\,k_2\,k_3}{9} + \frac{16\,\theta_i\,{k_3}^2}{9} + \frac{14\,{k_1}^2}{3} + 8\,k_1\,k_2 + \frac{48\,k_1\,k_3}{5} + \frac{64\,{k_2}^2}{15} \\
&+ \frac{104\,k_2\,k_3}{9} + \frac{2656\,{k_3}^2}{315} \le 0, \quad i = 1, \dots, N,
\end{aligned} \tag{57}$$

with $w = [k,\, y]^T$. Hence, by assuming $\epsilon = 10^{-2}$ and $\beta = 10^{-3}$, that corresponds according to formula (25) to $N = 2182$ samples. The solution of the scenario optimization problem (56)–(57) carried out using a genetic algorithm, is:

$$k_1^* = -0.6849,\ k_2^* = 2.5137,\ k_3^* = -1.3264 \text{ and } y^* = 0.5413, \tag{58}$$

that is:

$$u^*(t) = -0.6849 + 2.5137\,t - 1.3264\,t^2. \tag{59}$$

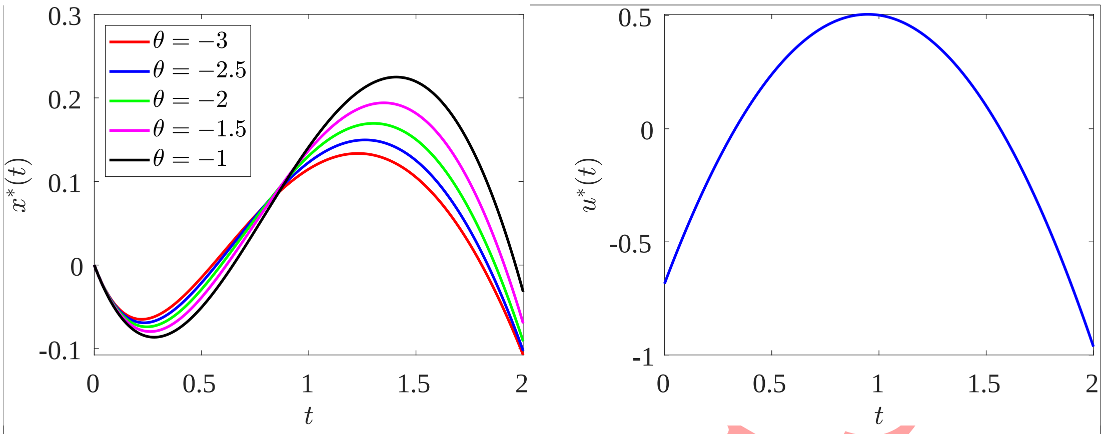
The optimal state trajectories for five samples of the parameter θ and the optimal control law are shown in Figures 1a and 1b, respectively. The corresponding values of the optimal performance index are reported in Table 1. Note also that the value of the performance index does not exceed the threshold value y^*. The mean value of the performance index is 0.3558, and the performance variation, defined as the difference between the worst-case and best-case values, is $\Delta J = 0.0227$. This demonstrates the robustness of the proposed control law.

Table 1: Example 1: Performance index values obtained for five uncertainty samples.

θ	-3	-2.5	-2	-1.5	-1
J^*	0.3465	0.3495	0.3538	0.3600	0.3692

Example 2.

Consider the following uncertain nonlinear optimal control problem:



(a) State **(b)** Control

Figure 1: Optimal Trajectories for Example 1.

$$\min_{u(t)} J(u(t)) = \int_0^1 x^2(t)\, dt \tag{60}$$

subject to:

$$\dot{x}(t) = \theta_1\, x^2(t) + \theta_2\, u(t), \tag{61}$$

$$x(0) = 0, \tag{62}$$

$$1 \leq \theta_1 \leq 2, \tag{63}$$

$$2 \leq \theta_2 \leq 3. \tag{64}$$

We seek a control law of the form:

$$u(t) = \sum_{i=1}^{2} k_i\, t^{i-1} = k_1 + k_2\, t. \tag{65}$$

Hence, by substituting the control (65) into the system (61), we obtain

$$\dot{x}(t) = \theta_1\, x^2(t) + \theta_2\, \left(k_1 + k_2\, t\right). \tag{66}$$

This equation can be solved iteratively using the following correction functional:

$$x^{(j+1)}(t) = x^{(j)}(t) + \int_0^t \lambda(s) \left(\dot{x}^{(j)}(s) - \theta_1 \left[\tilde{x}^{(j)}(s)\right]^2 - \theta_2\, \left(k_1 + k_2\, s\right) \right) ds. \tag{67}$$

The identification of the Lagrange multiplier using the calculus of variations [3, 18] yields $\lambda(t) = -1$. Consequently,

$$x^{(j+1)}(t) = x^{(j)}(t) - \int_0^t \dot{x}^{(j)}(s) - \theta_1 \left[x^{(j)}(s)\right]^2 - \theta_2\, \left(k_1 + k_2\, s\right)\, ds. \tag{68}$$

By performing two iterations, we obtain:

$$x(t) \approx x^{(2)}(t) = \frac{\theta_2\, t\, (2\, k_1 + k_2\, t)}{2} + \frac{\theta_1\, \theta_2^2\, t^3\, \left(20\, k_1^2 + 15\, k_1\, k_2\, t + 3\, k_2^2\, t^2\right)}{60}. \tag{69}$$

Then, by substituting the control (65) and the approximate solution (69) into the performance index (60), we obtain:

$$\begin{aligned}\tilde{J}(k,\, \theta) = {} & \frac{\theta_1^2\, \theta_2^4\, k_1^4}{63} + \frac{\theta_1^2\, \theta_2^4\, k_1^3\, k_2}{48} + \frac{23\, \theta_1^2\, \theta_2^4\, k_1^2\, k_2^2}{2160} + \frac{\theta_1^2\, \theta_2^4\, k_1\, k_2^3}{400} + \frac{\theta_1^2\, \theta_2^4\, k_2^4}{4400} \\ & + \frac{2\, \theta_1\, \theta_2^3\, k_1^3}{15} + \frac{5\, \theta_1\, \theta_2^3\, k_1^2\, k_2}{36} + \frac{\theta_1\, \theta_2^3\, k_1\, k_2^2}{20} + \frac{\theta_1\, \theta_2^3\, k_2^3}{160} + \frac{\theta_2^2\, k_1^2}{3} \\ & + \frac{\theta_2^2\, k_1\, k_2}{4} + \frac{\theta_2^2\, k_2^2}{20}. \end{aligned} \tag{70}$$

By assuming $\epsilon = 10^{-2}$ and $\beta = 10^{-3}$, which corresponds to $N = 1982$ samples, the solution of the corresponding scenario optimization problem (39)–(40), obtained using a genetic algorithm is:

$$k_1^* = 0.2830,\; k_2^* = -0.6993 \;\text{ and }\; y^* = 0.6130. \tag{71}$$

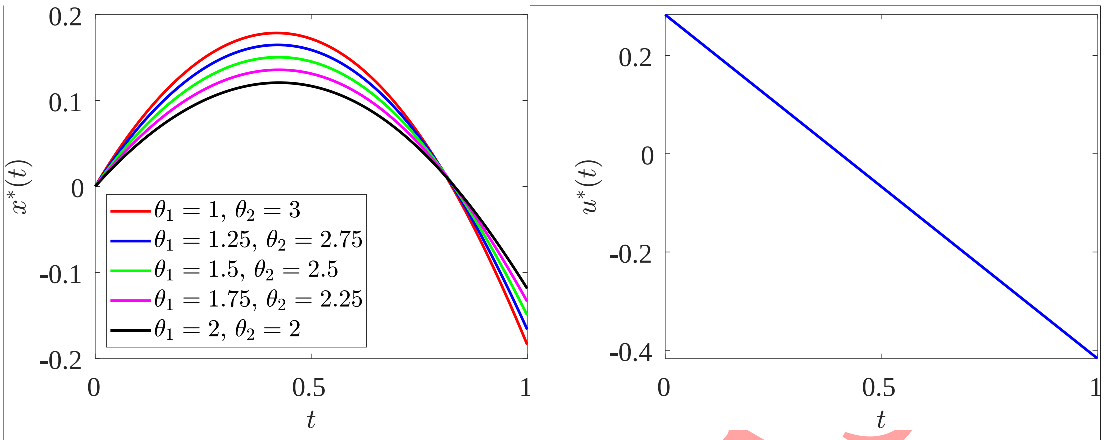
Therefore, the optimal control law is given by

$$u^*(t) = 0.2830 - 0.6993\, t. \tag{72}$$

The optimal trajectories of the state variables for five representative values of the parameter θ, together with the associated optimal control law, are illustrated in Figures 2a and 2b, respectively. The resulting values of the performance index are listed in Table 2. It can also be observed that the performance index remains below the prescribed threshold value y^*. The average value of the performance index is 0.0113, while the variation between the maximum and minimum performance values is $\Delta J = 0.0089$. These results confirm the robustness of the proposed control strategy.

Table 2: Example 2: Performance index values obtained for five uncertainty samples.

θ_1	1	1.25	1.5	1.75	1
θ_2	3	2.75	2.5	2.25	2
J^*	0.0159	0.0135	0.0112	0.009	0.007



(a) State **(b)** Control

Figure 2: Optimal Trajectories for Example 2.

6 Conclusion

This paper proposes a design approach for the optimal control of uncertain dynamical systems by integrating the parametric optimization method with the Variational Iteration Method (VIM) and the scenario approach. The control input is expressed as a finite polynomial expansion, and VIM is used to derive an approximate analytical solution of the state equation without discretization. Substituting this approximation into the performance index yields a robust min–max optimization problem, which is subsequently relaxed via probabilistic sampling into a scenario optimization problem solved using a genetic algorithm. The approach is validated on two scalar-state examples.

It should be noted that the accuracy of the VIM approximation is critical, and convergence for nonlinear or stiff systems requires verification of the contraction condition stated in Remark 1, which is not straightforward in practice. However, for small finite-time control problems, only a few iterations are typically required to achieve sufficient accuracy. Moreover, it is assumed that the probability distribution of the uncertainties is known, which is often not the case in practice; this limits the direct applicability of Theorem 1. In addition, small risk parameters lead to an optimization problem with a large number of constraints, which may render the solution computationally intractable.

The proposed design approach can be extended to other classes of fractional-order systems (currently under investigation), multi-state multi-input systems, and optimal control problems with constraints.

Declarations

Availability of Supporting Data

All data generated or analyzed during this study are included in this published article.

Funding

This research was conducted without external funding, grants, or financial support.

Conflict of Interest

The author declares no known competing financial interests or personal relationships that could have influenced the work reported in this paper.

Artificial Intelligence Statement

Artificial intelligence (AI) tools, including large language models, were used solely for language editing and improving readability. AI tools were not used for generating ideas, performing analyses, interpreting results, or writing the scientific content. All scientific conclusions and intellectual contributions were made exclusively by the authors.

Publisher's Note

The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[1] Akkouche, A., Maidi, A., Aidene, M. (2014). "Optimal control of partial differential equations based on the Variational Iteration Method". *Computers & Mathematics with Applications*, 68(5), 622–631. https://doi.org/10.1016/j.camwa.2014.07.007

[2] Allahdadi, M., Soradi-Zeid, S., Torabi, H., Hosseini, E. (2025). "Solving optimal control problems under interval uncertainty using robust model predictive control". *Iranian Journal of Fuzzy Systems*, 22(6), 147–165. https://doi.org/10.22111/ijfs.2025.52170.9204

[3] Berkani, S., Manseur, F., Maidi, A. (2012). "Optimal control based on the variational iteration method". *Computers & Mathematics with Applications*, 64(4), 604–610. https://doi.org/10.1016/j.camwa.2011.12.066

[4] Betts, J. T. (1998). "Survey of numerical methods for trajectory optimization". *Journal of Guidance, Control, and Dynamics*, 21(2), 193–207. https://doi.org/10.2514/2.4231

[5] Calafiore, G. C., Campi, M. C. (2006). "The scenario approach to robust control design". *IEEE Transactions on Automatic Control*, 51(5), 742–753. https://doi.org/10.1109/TAC.2006.8750411

[6] Campi, M. C., Garatti, S., Prandini, M. (2009). "The scenario approach for systems and control design". *Annual Reviews in Control*, 33(2), 149–157. https://doi.org/10.1016/j.arcontrol.2009.07.001

[7] Campi, M. C., Garatti, S. (2018). *Introduction to the Scenario Approach*. SIAM, Philadelphia. https://doi.org/10.1137/1.9781611975444

[8] Chakraverty, S., Mahato, N. R., Karunakar, P., Rao, T. D. (2019). *Advanced Numerical and Semi-Analytical Methods for Differential Equations*. John Wiley & Sons, New Jersey. https://doi.org/10.1002/9781119423461

[9] Choi, J., Deo, A., Lagoa, C., Subramanyam, A. (2024). "Reduced sample complexity in scenario-based control system design via constraint scaling". *IEEE Control Systems Letters*, 8, 2793–2798. https://doi.org/10.1109/LCSYS.2024.3515861

[10] Conway, B. A. (2012). "A survey of methods available for the numerical optimization of continuous dynamic systems". *Journal of Optimization Theory and Applications*, 152(2), 271–306. https://doi.org/10.1007/s10957-011-9918-z

[11] Corriou, J.-P. (2018). *Process Control: Theory and Applications*. Springer, London. https://doi.org/10.1007/978-3-319-61143-3

[12] Wu, D., Chen, Y., Yu, C., Bai, Y., Teo, K. L. (2023). "Control parameterization approach to time-delay optimal control problems: A survey". *Journal of Industrial and Management Optimization*, 19(5), 3750–3783. https://doi.org/10.3934/jimo.2022108

[13] Fikar, M., Latifi, M. A., Corriou, J. P., Creff, Y. (2000). "CVP-based optimal control of an industrial depropanizer column". *Computers & Chemical Engineering*, 24(2), 909–915. https://doi.org/10.1016/S0098-1354(00)00355-0

[14] Goh, C. J., Teo, K. L. (1988). "Control parametrization: A unified approach to optimal control problems with general constraints". *Automatica*, 24(1), 3–18. https://doi.org/10.1016/0005-1098(88)90003-9

[15] Goldberg, D. E. (1989). *Genetic Algorithms in Search, Optimization, and Machine Learning*. Addison-Wesley, Reading, MA.

[16] Hamdipoor, V., Nguyen, H. N., Mekhaldi, B., Parra, J., Badosa, J., Obaldia, F. C. (2025). "Experimental validation of scenario-based stochastic model predictive control of

nanogrids". *Control Engineering Practice*, 157, 106249. https://doi.org/10.1016/j.conengprac.2025.106249

[17] Hatamian, R., Hashemi, S. A. S. (2025). "A hybrid numerical approach for solving nonlinear optimal control problems". *Control and Optimization in Applied Mathematics*, 10(1), 125–138. https://doi.org/10.30473/coam.2025.72875.1273

[18] He, J.-H. (1998). "Approximate analytical solution for seepage flow with fractional derivatives in porous media". *Computer Methods in Applied Mechanics and Engineering*, 167(1), 57–68. https://doi.org/10.1016/S0045-7825(98)00108-X

[19] He, J.-H., Wu, X.-H. (2007). "Variational iteration method: New development and applications". *Computers & Mathematics with Applications*, 54(7), 881–894. https://doi.org/10.1016/j.camwa.2006.12.083

[20] Idiri, G., Djennoune, S., Bettayeb, M. (2016). "Solving fixed final time fractional optimal control problems using the parametric optimization method". *Asian Journal of Control*, 18(4), 1524–1536. https://doi.org/10.1002/asjc.1247

[21] Kustov, A. Y. (2023). "Parametrization of optimal anisotropic controllers". *Automation and Remote Control*, 10, 59–71. https://doi.org/10.1134/S0005117923100077

[22] Lauer, F. (2024). "Margin-based scenario approach to robust optimization in high dimension". *IEEE Transactions on Automatic Control*, 69(10), 7182–7189. https://doi.org/10.1109/TAC.2024.3393790

[23] Li, R., Teo, K. L., Wong, K. H., Duan, G. R. (2006). "Control parameterization enhancing transform for optimal control of switched system". *Mathematical and Computer Modelling*, 43(11), 1393–1403. https://doi.org/10.1016/j.mcm.2005.08.012

[24] Lichtenecker, D., Rixen, D., Eichmeir, P., Nachbagauer, K. (2023). "On the use of adjoint gradients for time-optimal control problems regarding a discrete control parameterization". *Multibody System Dynamics*, 59(3), 313–334. https://doi.org/10.1007/s11044-023-09898-5

[25] Lin, Q., Loxton, R., Teo, K. (2014). "The control parameterization method for nonlinear optimal control: A survey". *Journal of Industrial and Management Optimization*, 10(1), 275–309. https://doi.org/10.3934/jimo.2014.10.275

[26] Puschke, J., Djelassi, H., Kleinekorte, J., Hannemann-Tamás, R., Mitsos, A. (2018). "Robust dynamic optimization of batch processes under parametric uncertainty: Utilizing approaches from semi-infinite programs". *Computers & Chemical Engineering*, 116(9), 253–267. https://doi.org/10.1016/j.compchemeng.2018.05.025

[27] Rafiei, Z., Kafash, K., Karbassi, S. M. (2017). "A computational method for solving optimal control problems and their applications". *Control and Optimization in Applied Mathematics*, 2(1), 1–13. https://mathco.journals.pnu.ac.ir/article_4819.html

[28] Ramos, J. I. (2008). "On the variational iteration method and other iterative techniques for nonlinear differential equations". *Applied Mathematics and Computation*, 199(1), 39–69. https://doi.org/10.1016/j.amc.2007.09.024

[29] Rao, S. S. (2009). *Engineering Optimization: Theory and Practice*. John Wiley & Sons, New Jersey. https://doi.org/10.1002/9780470549124

[30] Rehbock, V., Teo, K. L., Jennings, L. S., Lee, H. W. J. (1999). "A survey of the control parametrization and control parametrization enhancing methods for constrained optimal control problems". *Progress in Optimization: Contributions from Australasia*, 30, 247–275. https://doi.org/10.1007/978-1-4613-3285-5_13

[31] Rocchetta, R., Mey, A., Oliehoek, F. A. (2024). "A survey on scenario theory, complexity, and compression-based learning and generalization". *IEEE Transactions on Neural Networks and Learning Systems*, 35(12), 16985–16999. https://doi.org/10.1109/TNNLS.2023.3308828

[32] Sargent, R. W. H. (2000). "Optimal control". *Journal of Computational and Applied Mathematics*, 124(1), 361–371. https://doi.org/10.1016/S0377-0427(00)00418-0

[33] Si, W., Gao, S., Zhang, M., Wen, T., Bai, Y., Wang, H. (2025). "Approximate optimal control for uncertain nonlinear systems: A reinforcement relearning framework". *Nonlinear Dynamics*, 113(18), 24821–24848. https://doi.org/10.1007/s11071-025-11393-9

[34] Stengel, R. F. (1994). *Optimal Control and Estimation*. Dover Publications, New York.

[35] Tatari, M., Dehghan, M. (2007). "On the convergence of He's variational iteration method". *Journal of Computational and Applied Mathematics*, 207(1), 121–128. https://doi.org/10.1016/j.cam.2006.07.017

[36] Teo, K. L., Goh, C. J., Wong, K. H. (1991). *A Unified Computational Approach to Optimal Control Problems*. Longman Scientific & Technical, New York.

[37] Teo, K. L., Jennings, L. S., Lee, H. W. J., Rehbock, V. (1999). "The control parameterization enhancing transform for constrained optimal control problems". *The Journal of the Australian Mathematical Society. Series B. Applied Mathematics*, 40, 314–335. https://doi.org/10.1017/S0334270000010936

[38] Terkmani, R., Maidi, A., Guermah, S., Aidene, M. (2020). "Receding horizon control based on the variational iteration method". *International Journal of Systems, Control and Communications*, 11(3), 242–260. https://doi.org/10.1504/IJSCC.2020.109072

[39] Trélat, E. (2012). "Optimal Control and Applications to Aerospace: Some Results and Challenges". *Journal of Optimization Theory and Applications*, 154(3), 713–758. https://doi.org/10.1007/s10957-012-0050-5

[40] Wadi, A., Vamvoudakis, K. G. (2026). "Trajectory-informed machine learning for quantum optimal control of uncertain systems". *Automatica*, 185, 112756. https://doi.org/10.1016/j.automatica.2025.112756

[41] Wang, X., Weyer, E. (2024). "Scenario based optimisation over uncertain system identification models". *Automatica*, 165, 111686. https://doi.org/10.1016/j.automatica.2024.111686

[42] Zagorowska, M., Falugi, P., O'Dwyer, E., Kerrigan, E. C. (2024). "Automatic scenario generation for efficient solution of robust optimal control problems". *International Journal of Robust and Nonlinear Control*, 34(2), 1370–1396. https://doi.org/10.1002/rnc.7038

[43] Zeidler, E. (1995). *Applied Functional Analysis: Applications to Mathematical Physics*. Springer, New York. https://doi.org/10.1007/978-1-4612-0815-0

[44] Zhang, L., Liu, C., Qian, C., Hua, C. (2026). "Prescribed-time optimal control of a class of uncertain nonlinear systems with unknown control direction". *Neurocomputing*, 672, 132683. https://doi.org/10.1016/j.neucom.2026.132683

Author Bio-sketch

Ghania Idiri received the Engineering degree in 2006, the Magister degree in 2011, and the Ph.D. degree in Automatic Control in 2017 from Mouloud Mammeri University of Tizi Ouzou. She is currently an Associate Professor with the Department of Automatic Control at Mouloud Mammeri University of Tizi Ouzou, where she teaches courses in system identification, optimization, and optimal control. Her research interests include optimization and optimal control of dynamic systems. Email: ghania.idiri@ummto.dz