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An Invex Active-Slater Test for Clarke-MFCQ in Nonsmooth Multiobjective Optimization

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Abstract. We consider a finite-dimensional multiobjective optimization problem with locally Lipschitz objective functions and finitely many locally Lipschitz inequality constraints. We formulate an active-set Slater test in which $\sigma_{\hat{x}}$ is selected from a structurally restricted admissible class or selection rule $\Sigma(\psi, \hat{x})$ prescribed before testing the qualification, rather than constructed from an MFCQ direction. Pointwise $(\sigma_{\hat{x}}, r_j)$-invex inequalities with $r_j \geq 0$ imply the Clarke–Mangasarian–Fromovitz constraint qualification (Clarke-MFCQ). Standard maximum-function and normal-cone arguments then yield normalized Clarke–KKT multipliers at locally weakly efficient solutions, positive objective multipliers for positive weighted-sum solutions, and a perturbed-KKT inclusion at locally isolated efficient solutions. If $\Sigma(\psi, \hat{x})$ contains every locally Lipschitz map, the test is equivalent to Clarke-MFCQ. A nonsmooth nonconvex example illustrates the assumptions and multiplier inclusions.

Keywords. Nonsmooth multiobjective optimization, Clarke subdifferential, constraint qualification, Invexity, KKT conditions, Isolated efficient solution.

MSC. 49J52; 90C29; 90C30.

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1 Introduction

Multiobjective models with locally Lipschitz data arise when several objectives must be optimized simultaneously and differentiability is unavailable ([6, 32]). In this setting, first-order necessary conditions are naturally expressed through Clarke subdifferentials and require a constraint qualification to exclude abnormal Fritz–John multipliers; see, e.g., [3].

In this paper, we study

$$(\mathcal{MP}): \quad \min \quad \phi(x) := \big(\phi_1(x), \dots, \phi_p(x)\big),$$
$$\text{subject to} \quad \psi_j(x) \leq 0, \qquad j \in J := \{1, \dots, m\}.$$

where every objective $\phi_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I := \{1, \dots, p\}$, and every constraint $\psi_j : \mathbb{R}^n \to \mathbb{R}$, $j \in J$, is locally Lipschitz.

When $p = 1$, the problem $(\mathcal{MP})$ reduces to the classical nonsmooth optimization problem, hereafter referred to as OP. Two types of necessary optimality conditions are of particular interest in the context of OP: the Fritz–John (FJ) conditions and the Karush-Kuhn-Tucker (KKT) conditions. For further details, we refer the reader to [7] for the differentiable case, [12] for the nondifferentiable convex case, [17] for the quasiconvex case, and [3, 29] for the nondifferentiable nonconvex case.

Necessary optimality conditions play a fundamental role in characterizing solutions to MOPs and in guiding the development of solution algorithms. In the context of nonsmooth multiobjective problems, classical differentiability-based conditions, such as the KKT conditions, are no longer directly applicable. To address this limitation, generalized concepts of subdifferentials and constraint qualifications must be employed to extend optimality conditions to the nonsmooth setting. Depending on the regularity properties of the objective and constraint functions—whether they are differentiable, convex, locally Lipschitz, or lower semicontinuous—the corresponding optimality conditions are formulated using their gradient, convex subdifferential, Clarke subdifferential, or limiting subdifferential, respectively; see, for example, [7, 23, 32]. In this work, we present our main results using the Clarke subdifferential framework.

Nonsmooth invexity and its applications to multiobjective optimization are well established in the literature; see, for example, [1, 19, 22, 25, 26]. Building on this literature, we formulate an active-set Slater-type condition through a prescribed, structurally restricted admissible class of maps $\Sigma(\psi, \hat{x})$. We prove that this condition yields a common strict Clarke descent direction for all active constraints and therefore implies Clarke-MFCQ. This connection leads to normalized KKT necessary conditions for weakly efficient and positive weighted-sum solutions, together with a perturbed-KKT necessary condition for isolated efficient solutions. We also clarify the role of the admissible map class. If all locally Lipschitz maps are permitted, the zero-parameter form of the proposed condition is equivalent to Clarke-MFCQ. Thus, the condition has independent content only when the admissible class is specified by a meaningful structural restriction before the qualification is tested. We derive multiplier conditions for the following four types of optimal solutions of $(\mathcal{MP})$:

- Locally weakly efficient solutions;
- Locally efficient solutions;
- Local minimizers of positive weighted-sum scalarizations;
- Locally isolated efficient solutions.

The precise definitions are given in Section 2.

Weakly efficient and efficient solutions have been characterized in [16] (resp. [33], [7], and [5, 14, 18, 31]) for linear (resp. convex, smooth, and nonsmooth) MOPs. Invexity has long been used to derive optimality and duality results beyond convexity. Locally Lipschitz r-invex functions and Slater-type conditions were studied by Antczak [1], and invex nonsmooth multiobjective problems were treated by Kim and Schaible [19]. Weak Slater and strong KKT conditions for nonsmooth multiobjective semi-infinite problems were developed by Habibi, Kanzi, and Ebadian [10] and by Kanzi [15]. For isolated efficient solutions, Goberna and Kanzi [8] introduced a perturbed-KKT condition in the convex semi-infinite setting, and related nonsmooth characterizations were obtained by Rezaee [27]. Beyond the Euclidean and semi-infinite settings considered here, constraint qualifications and KKT-type conditions for nonsmooth multiobjective and equilibrium-constrained problems continue to be actively developed, e.g., via convexificators [20, 28, 30] and on Hadamard manifolds [34]. The present paper considers finitely many locally Lipschitz constraints in $\mathbb{R}^n$ and derives these consequences from an active invex-Slater sufficient condition for Clarke-MFCQ, without appealing to convexificators or a manifold structure.

The paper makes three claims. First, the active invex-Slater condition implies Clarke-MFCQ. Second, a maximum-function argument gives Fritz–John and normalized KKT conditions at locally weakly efficient points; efficient points follow as a corollary, and positive weighted-sum solutions yield positive objective multipliers. Third, locally isolated efficient points satisfy a perturbed-KKT neighborhood inclusion. Section 2 collects the required nonsmooth tools, introduces the active condition, and gives a nonconvex example used throughout the paper. Section 3 establishes the multiplier results for weakly efficient, efficient, positive weighted-sum, and isolated efficient solutions, and Section 4 concludes.

2 Preliminaries

We briefly overview some notions of variational analysis in this section. For more details, examples, discussion, and applications see [3, 12, 29].

Suppose that 0_n and $\mathbb{B}_n$ denote the zero-vector of $\mathbb{R}^n$ and the closed unit ball in $\mathbb{R}^n$, respectively, i.e.,

$$0_n := (\overbrace{0,\ldots,0}^{n\ times}) \qquad \text{and} \qquad \mathbb{B}_n := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}.$$

As usual, $\langle x, y\rangle$ denotes the standard inner product of $x, y \in \mathbb{R}^n$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The generalized Clarke directional derivative of f at $\hat{x} \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$, and the Clarke subdifferential of f at $\hat{x}$ are respectively defined as

$$f^0(\hat{x}; d) := \limsup_{y\to\hat{x}\ t\downarrow 0} \frac{f(y+td) - f(y)}{t},$$

$$\partial_c f(\hat{x}) := \{\xi \in \mathbb{R}^n \mid \ f^0(\hat{x}; d) \geq \langle \xi, d\rangle, \quad \forall d \in \mathbb{R}^n\}.$$

Observe that, if $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable at $\hat{x}$, then $\partial_c f(\hat{x}) = \{\nabla f(\hat{x})\}$, where $\nabla f(\hat{x})$ denotes the classical gradient of f at $\hat{x}$. Also, if $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function, then it is locally Lipschitz, $\phi^0(\hat{x}; d) = \phi'(\hat{x}; d)$, and $\partial_c f(\hat{x}) = \partial f(\hat{x})$, in which $f'(\hat{x}; d)$ and $\partial f(\hat{x})$ denote the standard directional derivative and the convex subdifferential of f, defined respectively as

$$f'(\hat{x}; d) := \lim_{t\downarrow 0} \frac{f(\hat{x}+td) - f(\hat{x})}{t},$$

$$\partial f(\hat{x}) := \{\xi \in \mathbb{R}^n \mid f(x) - f(\hat{x}) \geq \langle \xi, x - \hat{x}\rangle\,,\ \forall x \in \mathbb{R}^n\}\,. \tag{1}$$

Theorem 1. [3] Let f and g be locally Lipschitz functions from $\mathbb{R}^n$ to $\mathbb{R}$ and $\hat{x} \in \mathbb{R}^n$ be given. Then,

(i) $f^0(\hat{x}; \cdot)$ is a convex function and $\partial_c f(\hat{x}) = \partial f^0(\hat{x}; \cdot)(0_n)$.

(ii) $\partial_c(\alpha f + g)(\hat{x}) \subseteq \alpha\partial_c f(\hat{x}) + \partial_c g(\hat{x}), \qquad \forall \alpha \in \mathbb{R}.$

(iii) $f^0(\hat{x}; d) = \max_{\xi\in\partial_c f(\hat{x})} \langle \xi, d\rangle, \qquad \forall d \in \mathbb{R}^n.$

Given a nonempty set $A \subseteq \mathbb{R}^n$, we denote by $int(A)$, $\overline{A}$, $conv(A)$, and $cone(A)$, the interior of A, the closure of A, the convex hull of A, and the convex cone (containing the origin) generated by A, respectively.

Theorem 2. [29] Let $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz functions, and let

$$F(x) := \max\Big\{f_\ell(x) \mid \ell \in L := \{1,\ldots,k\}\Big\}, \qquad \forall x \in \mathbb{R}^n.$$

Then, $F(\cdot)$ is a locally Lipschitz function and

$$\partial_c F(\hat{x}) \subseteq conv\left(\bigcup_{\ell\in L^*_{\hat{x}}} \partial_c f_\ell(\hat{x})\right) \subseteq conv\left(\bigcup_{\ell\in L} \partial_c f_\ell(\hat{x})\right), \qquad \forall \hat{x}\in\mathbb{R}^n$$

where $L^*_{\hat{x}}$ defined as

$$L^*_{\hat{x}} := \{\ell\in L \mid f_\ell(\hat{x}) = F(\hat{x})\}.$$

Theorem 3. [12] Let $A_1,\ldots,A_k$ be nonempty convex subsets of $\mathbb{R}^n$. Then

$$conv\left(\bigcup_{1\le\ell\le k} A_\ell\right) = \left\{\sum_{\ell=1}^k \beta_\ell a_\ell \mid \sum_{\ell=1}^k \beta_\ell = 1,\ \ a_\ell\in A_\ell \text{ and } \beta_\ell\ge 0 \text{ for } 1\le\ell\le k\right\},$$

$$cone\left(\bigcup_{1\le\ell\le k} A_\ell\right) = \left\{\sum_{\ell=1}^k \beta_\ell a_\ell \mid\ a_\ell\in A_\ell \text{ and } \beta_\ell\ge 0 \text{ for } 1\le\ell\le k\right\}.$$

Here, zero coefficients β_ℓ allow arbitrary choices of a_ℓ. If, in addition, each A_ℓ is compact and $0_n\notin conv\big(\bigcup_\ell A_\ell\big)$, then $cone\big(\bigcup_\ell A_\ell\big)$ is closed.

The polar and the strictly polar of $A\subseteq\mathbb{R}^n$ are, respectively, defined by

$$A^{\leq} := \{x\in\mathbb{R}^n \mid \langle x,a\rangle \le 0,\ \ \forall a\in A\},$$

$$A^{<} := \{x\in\mathbb{R}^n \mid \langle x,a\rangle < 0,\quad \forall a\in A\}.$$

Notice that $A^{\leq}$ is always a closed convex cone, and

$$A^{<}\neq\emptyset \implies A^{\leq} = \overline{A^{<}}.$$

Theorem 4. [29] Suppose that A_1 and A_2 are two subsets of $\mathbb{R}^n$. One has

$$A_1\subseteq A_2 \implies A_2^{\leq}\subseteq A_1^{\leq} \implies \overline{cone}(A_1)\subseteq\overline{cone}(A_2).$$

Theorem 5. [12] Let the non-empty compact set $A\subseteq\mathbb{R}^n$ be given. Then, $conv(A)$ is a closed set. Also, $cone(A)$ is closed if $0_n\notin conv(A)$, where

Let the nonempty closed set $A\subseteq\mathbb{R}^n$ be given. The Clarke tangent cone and the Clarke normal cone of A at $\hat{x}\in A$ are, respectively, defined by

$$\Gamma(A,\hat{x}) := \left\{d\in\mathbb{R}^n \mid \forall x_r\to\hat{x} \text{ in } A,\ \forall t_r\downarrow 0,\ \exists d_r\to d,\ x_r+t_rd_r\in A,\ \ \forall r\in\mathbb{N}\right\},$$

and

$$N(A,\hat{x}) := (\Gamma(A,\hat{x}))^{\leq}.$$

Both $\Gamma(A,\hat{x})$ and $N(A,\hat{x})$ are always closed convex cones; see [3, 29].

Theorem 6. [29] Let $A \subseteq \mathbb{R}^n$ be closed, $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function, and $h : \mathbb{R}^n \to \mathbb{R}$ be a convex function. If $\hat{x} \in A$ is a local minimum point of $f - h$ over A,then

$$\partial h(\hat{x}) \subseteq \partial_c f(\hat{x}) + N(A, \hat{x}).$$

The following result is a direct corollary of the above theorem, by taking $h = 0$.

Theorem 7. [29] Let $A \subseteq \mathbb{R}^n$ be closed. If the locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ attains a local minimum on A at the point $\hat{x} \in A$, then

$$0_n \in \partial_c f(\hat{x}) + N(A, \hat{x}).$$

A short justification of Theorem 6 from Theorem 7 is worth recording, since it is used in Section 3. If $v \in \partial h(\hat{x})$, then the convex subgradient inequality gives $h(x) - h(\hat{x}) \geq \langle v, x - \hat{x} \rangle$ for every $x \in \mathbb{R}^n$, so $x \mapsto f(x) - \langle v, x \rangle$ attains a local minimum over A at $\hat{x}$ whenever $f - h$ does. Applying Theorem 7 to $f(\cdot) - \langle v, \cdot \rangle$ and using $\partial_c\big(f - \langle v, \cdot \rangle\big)(\hat{x}) = \partial_c f(\hat{x}) - v$ (Theorem 1(ii)) yields $v \in \partial_c f(\hat{x}) + N(A, \hat{x})$, which proves Theorem 6 since $v \in \partial h(\hat{x})$ was arbitrary.

3 Necessary Optimality Conditions

In this section we consider the following MOP:

$$(\mathcal{MP}): \qquad \min \phi(x) := \big(\phi_1(x), \ldots, \phi_p(x)\big) \quad \text{s.t.} \quad x \in \mathcal{S},$$

where $\phi_i : \mathbb{R}^n \to \mathbb{R}$, for $i \in I := \{1, \ldots, p\}$, are locally Lipschitz functions and the feasible set $\mathcal{S}$ is defined as

$$\mathcal{S} := \{x \in \mathbb{R}^n \mid \psi_j(x) \leq 0, \quad \forall j \in J\}, \tag{2}$$

in which $J := \{1, \ldots, m\}$ and the constraint function ψ_j, for $j \in J$, are locally Lipschitz from $\mathbb{R}^n$ to $\mathbb{R}$.

The next definition summarizes several commonly used notions of optimality for the multiobjective problem $(\mathcal{MP})$.

Definition 1. Let $\hat{x} \in \mathcal{S}$ be a feasible point. It is classified as follows:

- $\hat{x}$ is a *locally weakly efficient solution* of $(\mathcal{MP})$ if there exists a neighborhood U of $\hat{x}$ such that no $x \in \mathcal{S} \cap U$ satisfies

$$\phi_i(x) < \phi_i(\hat{x}) \quad \text{for every } i \in I.$$

- $\hat{x}$ is a *locally efficient solution* of $(\mathcal{MP})$ if there exists a neighborhood U of $\hat{x}$ such that no $x \in \mathcal{S} \cap U$ satisfies

$$\phi_i(x) \leq \phi_i(\hat{x}) \quad \text{for all } i \in I, \qquad \phi_{i_0}(x) < \phi_{i_0}(\hat{x}) \quad \text{for some } i_0 \in I.$$

- $\hat{x}$ is a *positive weighted-sum solution* of $(\mathcal{MP})$ if there exist positive weights $\lambda_i > 0$, $i \in I$, for which $\hat{x}$ locally minimizes the corresponding weighted sum over $\mathcal{S}$, i.e., there is a neighborhood U of $\hat{x}$ such that

$$\sum_{i=1}^{p} \lambda_i \phi_i(\hat{x}) \leq \sum_{i=1}^{p} \lambda_i \phi_i(x) \quad \text{for all } x \in \mathcal{S} \cap U.$$

- $\hat{x}$ is a *locally isolated efficient solution* with constant $\nu > 0$ if there exists a neighborhood U of $\hat{x}$ such that

$$\max_{1 \leq i \leq p} \{\phi_i(x) - \phi_i(\hat{x})\} \geq \nu \|x - \hat{x}\| \qquad \text{for all } x \in \mathcal{S} \cap U.$$

The corresponding global notions are recovered by taking $U = \mathbb{R}^n$.

Remark 1. **(i)** Every locally isolated efficient point is locally efficient, every locally efficient point is locally weakly efficient, and every positive weighted-sum solution is locally efficient:

$$\text{Locally Isolated Efficient} \implies \text{Locally Efficient} \implies \text{Locally Weakly Efficient},$$

$$\text{Positive Weighted-Sum Solution} \implies \text{Locally Efficient}.$$

(ii) The notions of weak efficiency and efficiency can be interpreted via partial orders in the objective space $\mathbb{R}^p$, generated, respectively, by the nonnegative and positive orthants

$$\mathbb{R}^p_+ := \{(y_1, \ldots, y_p) \in \mathbb{R}^p \mid y_i \geq 0, \ \forall i \in I\},$$

and

$$\mathbb{R}^p_{++} := \{(y_1, \ldots, y_p) \in \mathbb{R}^p \mid y_i > 0, \ \forall i \in I\}.$$

(iii) Positive weighted-sum scalarization is a widely used technique for converting a vector optimization problem into an equivalent scalar one, allowing the application of classical scalar optimization tools; see [6] for an extensive overview. The resulting solution concept is closely related to, but should not be confused with, the several standard notions of proper efficiency (Geoffrion, Benson, Borwein, Henig); see [6, 9] for a comparison among these notions.

(iv) Isolated efficiency is stable under sufficiently small Lipschitz perturbations of the objective components: if $\hat{x}$ is a locally isolated efficient solution of $(\mathcal{MP})$ with isolation constant $\nu > 0$, and each $u^i \in \mathbb{R}^n$, $i \in I$, satisfies $\|u^i\| < \nu$, then $\hat{x}$ remains locally isolated efficient, with reduced constant $\nu - \tau$ where $\tau := \max_{i \in I} \|u^i\|$, for the linearly perturbed problem

$$(\mathcal{MP}_u): \qquad \min\ \left(\phi_1(x) + \langle u^1, x\rangle, \ldots, \phi_p(x) + \langle u^p, x\rangle\right)$$
$$\text{subject to} \quad \psi_j(x) \leq 0, \qquad j \in J.$$

This is a direct consequence of the triangle inequality applied to $\max_{i\in I}\{(\phi_i+\langle u^i,\cdot\rangle)(x)-(\phi_i+\langle u^i,\cdot\rangle)(\hat{x})\}$, and it motivates the perturbed-KKT condition established in Theorem 12; see also [8, 16, 27] for related sensitivity results in the scalar and semi-infinite settings.

We now recall and slightly refine, for a fixed base point, a class of generalized convexity for locally Lipschitz functions that goes back to Reiland [25, 26], Lee [22], Antczak [1], Vial [35], and Kim and Schaible [19].

Definition 2. Let $\hat{x} \in \mathbb{R}^n$, let $\sigma_{\hat{x}} : \mathbb{R}^n \to \mathbb{R}^n$ be a given mapping, and let $r \in \mathbb{R}$ be fixed. A locally Lipschitz function $\vartheta : \mathbb{R}^n \to \mathbb{R}$ is said to be $(\sigma_{\hat{x}}, r)$*-invex* at $\hat{x}$ if, for every $x \in \mathbb{R}^n$, the inequality

$$\vartheta(x) - \vartheta(\hat{x}) \geq \langle \xi, \sigma_{\hat{x}}(x)\rangle + r\|x - \hat{x}\|^2, \quad \text{for all } \xi \in \partial_c\vartheta(\hat{x}) \tag{3}$$

is satisfied. The function ϑ is called *strongly* r*-invex* and *weakly* r*-invex* at $\hat{x}$ whenever $r > 0$ and $r < 0$, respectively.

Remark 2. Several known concepts arise as particular instances of Definition 2:

- If $r = 0$ and $\sigma_{\hat{x}}(x) = x - \hat{x}$, then the $(\sigma_{\hat{x}}, r)$-invex inequality (3) reduces to the usual convex subdifferential inequality (1).
- Setting $r = 0$ yields the concept of σ-invexity; the differentiable and nonsmooth formulations can be found in [11] and [19], respectively. Moreover, for a function $\vartheta : \mathbb{R}^n \to \mathbb{R}$, r-convexity at $\hat{x}$ is a stronger (respectively, weaker) condition than convexity when $r > 0$ (respectively, $r < 0$).
- If $r = 0$ and $\sigma_{\hat{x}}(x) = \dfrac{\rho(x,\hat{x})}{b(x,\hat{x})}$, where $\rho : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)^n$ and $b : \mathbb{R}^n \times \mathbb{R}^n \to (0, +\infty)$, then Definition 2 reduces to the notion of b_ρ-invexity introduced in [21].
- In the particular case $\sigma_{\hat{x}}(x) = x - \hat{x}$, the r-invex condition becomes the nonsmooth r-convexity concept proposed by Vial [35].
- The quadratic penalty term $\|x - \hat{x}\|^2$ may be replaced by $\|\theta(x, \hat{x})\|^2$ for a suitable vector-valued mapping $\theta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, as in the more general framework of Jeyakumar [13].

For each $\hat{x} \in \mathcal{S}$, the set of all active constraint indices at $\hat{x}$ is denoted by $J(\hat{x})$ and is defined as

$$J(\hat{x}) := \{j \in J \mid \psi_j(\hat{x}) = 0\}.$$

Also, we define the following subdifferential sets:

$$\Phi_{\hat{x}} := \bigcup_{i \in I} \partial_c \phi_i(\hat{x}) \quad \text{and} \quad \Psi_{\hat{x}} := \bigcup_{j \in J(\hat{x})} \partial_c \psi_j(\hat{x}), \qquad \forall \hat{x} \in \mathcal{S}.$$

The following definition is recalled from [10, 12].

Definition 3. The Slater Constraint Qualification (SCQ) is said to hold for the problem $(\mathcal{MP})$ if the constraint function ψ_j, for each $j \in J$, is convex, and there exists a point $x^* \in \mathbb{R}^n$, called a Slater point, such that

$$\psi_j(x^*) < 0 \quad \text{for all } j \in J.$$

The following definition, introduced here for the first time, generalizes the classical Slater Constraint Qualification (SCQ) and is motivated by the preceding discussion. A central issue, addressed immediately below, is that the map $\sigma_{\hat{x}}$ used in the invex inequality cannot be an unrestricted, freely chosen object: its status as a genuine constraint qualification depends entirely on how it is quantified.

Definition 4 (Active (σ, r)-Slater condition)**.** Fix a feasible point $\hat{x} \in \mathcal{S}$. A *structurally restricted admissible class or selection rule* $\Sigma(\psi, \hat{x})$ of maps $\sigma_{\hat{x}} : \mathbb{R}^n \to \mathbb{R}^n$ is prescribed *before* the qualification is tested; the rule may depend on the constraint data ψ, but it must not be defined using a direction already known to satisfy Clarke-MFCQ at $\hat{x}$. For $\sigma_{\hat{x}} \in \Sigma(\psi, \hat{x})$ and real numbers $r_j \geq 0$, $j \in J(\hat{x})$, we write

$$A(\hat{x}; \sigma_{\hat{x}}, r)$$

if every active constraint ψ_j, $j \in J(\hat{x})$, is $(\sigma_{\hat{x}}, r_j)$-invex at $\hat{x}$, *and* there exists $x^s \in \mathbb{R}^n$, called an *active Slater point*, such that

$$\psi_j(x^s) < 0 \quad \text{for every } j \in J(\hat{x}).$$

At the class level, we write

$$A_\Sigma(\hat{x}; r) \;:\Longleftrightarrow\; \exists\, \sigma_{\hat{x}} \in \Sigma(\psi, \hat{x}) \text{ such that } A(\hat{x}; \sigma_{\hat{x}}, r).$$

We say that $(\mathcal{MP})$ fulfills the σ_r-invex Slater constraint qualification (σ_r-Invex-SCQ) at $\hat{x}$ if $A_\Sigma(\hat{x}; r)$ holds for the prescribed class $\Sigma(\psi, \hat{x})$. The active Slater point x^s is required to strictly satisfy only the constraints active at $\hat{x}$; it need not be feasible for the full system (2).

Quantification of $\sigma_{\hat{x}}$ and an unrestricted converse

The status of Definition 4 as a genuinely new hypothesis hinges on how $\Sigma(\psi, \hat{x})$ is specified. For *every* prescribed admissible class and every $\sigma_{\hat{x}} \in \Sigma(\psi, \hat{x})$, the map-specific condition implies Clarke-MFCQ at $\hat{x}$ (Proposition 1 below); equivalently, $A_\Sigma(\hat{x}; r) \Rightarrow$ Clarke-MFCQ$(\hat{x})$. The converse cannot hold for an arbitrary class: if $\Sigma(\psi, \hat{x})$ is allowed to contain *every* locally

Lipschitz map $\mathbb{R}^n \to \mathbb{R}^n$, then the zero-parameter form of the condition is equivalent to Clarke-MFCQ.

Indeed, suppose Clarke-MFCQ holds at $\hat{x}$, i.e., $J(\hat{x}) \neq \emptyset$ and there is $d \in \mathbb{R}^n$ with $\psi_j^0(\hat{x}; d) < 0$ for every $j \in J(\hat{x})$; write $a_j := \psi_j^0(\hat{x}; d) < 0$. For sufficiently small $t > 0$, $x^s := \hat{x} + td$ strictly satisfies every active constraint. Define

$$\tau(x) := \max\Big(0, \max_{j \in J(\hat{x})} \frac{\psi_j(x) - \psi_j(\hat{x})}{a_j}\Big), \qquad \sigma_{\hat{x}}(x) := \tau(x)\, d.$$

For every $\xi \in \partial_c \psi_j(\hat{x})$ and $j \in J(\hat{x})$, Theorem 1(iii) gives $\langle \xi, d\rangle \leq a_j$, and $\tau(x) \geq 0$, so

$$\langle \xi, \sigma_{\hat{x}}(x)\rangle = \tau(x)\langle \xi, d\rangle \leq \tau(x) a_j \leq \psi_j(x) - \psi_j(\hat{x}).$$

Hence each active constraint is $(\sigma_{\hat{x}}, 0)$-invex at $\hat{x}$. Since τ is a finite maximum of locally Lipschitz functions (Theorem 2), τ and hence $\sigma_{\hat{x}}$ are locally Lipschitz, so $\sigma_{\hat{x}}$ belongs to the unrestricted class. This proves

$$\text{Clarke-MFCQ}(\hat{x}) \iff A_\Sigma(\hat{x}; 0) \quad \text{when } \Sigma(\psi, \hat{x}) \text{ contains every locally Lipschitz map.}$$

Consequently, continuity or local Lipschitz continuity of $\sigma_{\hat{x}}$ does not by itself confer novelty on the condition: the map constructed above already has that regularity. The condition has independent content, beyond a restatement of Clarke-MFCQ, only when $\Sigma(\psi, \hat{x})$ is specified by a genuine structural restriction fixed in advance of testing the qualification. We record this dependence explicitly rather than leaving the quantifier on σ implicit.

We also record that the global invexity hypothesis in Definition 4 is used, in every proof below, only through the one-point estimate at the active Slater point x^s,

$$\langle \xi, \sigma_{\hat{x}}(x^s)\rangle + r_j\|x^s - \hat{x}\|^2 \leq \psi_j(x^s) - \psi_j(\hat{x}) < 0, \qquad \xi \in \partial_c \psi_j(\hat{x}),\ j \in J(\hat{x});$$

no theorem in this paper uses the invex inequality at any point other than x^s. A minimal sufficient certificate for all results below is therefore the existence of x^s and $d \in \mathbb{R}^n$ with

$$\psi_j(x^s) < 0, \qquad \psi_j^0(\hat{x}; d) < 0, \qquad \forall j \in J(\hat{x}),$$

i.e., Clarke-MFCQ itself is the operative hypothesis; the invexity language is retained because it produces the required direction d constructively from x^s via Proposition 1.

Example 1. Consider

$$\min\ \big(\phi_1(x), \phi_2(x)\big) := \big(|x|, -2x\big) \qquad \text{subject to} \qquad \psi(x) := x + x^2 + 3(1 - \cos x) \leq 0.$$

The objective ϕ_1 is genuinely nonsmooth, and the constraint is nonconvex because $\psi''(\pi) = 2 + 3\cos\pi = -1 < 0$. At $\hat{x} = 0$,

$$\psi(0) = 0, \qquad \partial_c\psi(0) = \{1\}, \qquad \psi(x) - \psi(0) = x + x^2 + 3(1 - \cos x) \geq x \quad \forall x \in \mathbb{R},$$

so ψ is $(\sigma_0, 0)$-invex at 0 for the prescribed map $\sigma_0(x) := x$. Moreover,

$$\psi(-1/4) \leq -\tfrac{1}{4} + \tfrac{1}{16} + \tfrac{3}{32} = -\tfrac{3}{32} < 0,$$

so $x^s = -1/4$ is an active Slater point and the σ_r-Invex-SCQ holds at $\hat{x} = 0$ with $J(0) = \{1\}$ (writing the single constraint as index 1) and $r_1 = 0$. Every feasible point satisfies $x \leq 0$, so

$$\max\{\phi_1(x) - \phi_1(0),\, \phi_2(x) - \phi_2(0)\} = \max\{|x|, -2x\} = 2|x| \quad \forall x \leq 0,$$

and hence $\hat{x} = 0$ is a globally isolated efficient solution with constant $\nu = 2$. Also, at 0,

$$\partial_c\phi_1(0) = [-1, 1], \qquad \partial_c\phi_2(0) = \{-2\}, \qquad \partial_c\psi(0) = \{1\}.$$

This example is used throughout the paper to illustrate the multiplier inclusions obtained below.

We now establish the proposition that is the conceptual center of the paper: the active (σ, r)-Slater condition is a sufficient condition for Clarke-MFCQ.

Proposition 1 (Implication to Clarke-MFCQ)**.** If the active (σ, r)-Slater condition holds at $\hat{x} \in \mathcal{S}$ with $r_j \geq 0$ for all $j \in J(\hat{x})$, then there exists $d \in \mathbb{R}^n$ such that

$$\psi_j^0(\hat{x}; d) < 0 \qquad \forall j \in J(\hat{x}).$$

Hence the active condition is a sufficient condition for Clarke-MFCQ at $\hat{x}$.

Proof. Set $d := \sigma_{\hat{x}}(x^s)$, where x^s is the active Slater point. For $j \in J(\hat{x})$ and $\xi \in \partial_c\psi_j(\hat{x})$, the $(\sigma_{\hat{x}}, r_j)$-invexity of ψ_j at $\hat{x}$ gives

$$\langle \xi, d \rangle \leq \psi_j(x^s) - \psi_j(\hat{x}) - r_j\|x^s - \hat{x}\|^2 = \psi_j(x^s) - r_j\|x^s - \hat{x}\|^2 < 0,$$

since $\psi_j(\hat{x}) = 0$, $\psi_j(x^s) < 0$, and $r_j \geq 0$. Taking the maximum over $\xi \in \partial_c\psi_j(\hat{x})$ and using Theorem 1(iii) gives $\psi_j^0(\hat{x}; d) < 0$. $\square$

We first record the Fritz–John necessary condition, which requires no constraint qualification, and then combine it with Proposition 1 to obtain the normalized KKT condition.

Theorem 8 (Fritz–John Condition at Locally Weakly Efficient Solutions)**.** If $\hat{x}$ is a locally weakly efficient solution of $(\mathcal{MP})$, then there exist coefficients $\alpha_i \geq 0$, $i \in I$, and $\beta_j \geq 0$, $j \in J(\hat{x})$, not all zero, such that

$$\sum_{i \in I} \alpha_i + \sum_{j \in J(\hat{x})} \beta_j = 1 \tag{4}$$

and

$$0_n \in \sum_{i \in I} \alpha_i \partial_c\phi_i(\hat{x}) + \sum_{j \in J(\hat{x})} \beta_j \partial_c\psi_j(\hat{x}). \tag{5}$$

Proof. Let U be a neighborhood of $\hat{x}$ witnessing local weak efficiency, and consider the locally Lipschitz function

$$\Theta(x) := \max\left\{\max_{1\le i\le p}\{\phi_i(x)-\phi_i(\hat{x})\}\,,\ \max_{j\in J}\{\psi_j(x)\}\right\}.$$

If $x \in \mathcal{S}\cap U$, then $\phi_i(x)-\phi_i(\hat{x}) \ge 0$ for some $i \in I$ by local weak efficiency of $\hat{x}$, so $\Theta(x) \ge 0$; if $x \notin \mathcal{S}$, then $\psi_j(x) \ge 0$ for some $j \in J$, so $\Theta(x) \ge 0$ as well. Hence $\Theta(x) \ge 0 = \Theta(\hat{x})$ for every $x \in U$, and $\hat{x}$ is a local minimizer of Θ (over $\mathbb{R}^n$, with U as the localizing neighborhood). By Theorem 7 applied with $A = \mathbb{R}^n$ and $N(\mathbb{R}^n, \hat{x}) = \{0_n\}$, together with Theorem 2,

$$0_n \in \partial_c\Theta(\hat{x}) \subseteq conv\left(\bigcup_{i\in I}\partial_c\phi_i(\hat{x}) \cup \bigcup_{j\in J(\hat{x})}\partial_c\psi_j(\hat{x})\right) = conv(\Phi_{\hat{x}} \cup \Psi_{\hat{x}}), \tag{6}$$

where the active set replaces J because $\psi_j(\hat{x}) < 0$ for $j \notin J(\hat{x})$ makes those indices non-maximizing near $\hat{x}$. Theorem 3 now yields (4) and (5), with $\alpha_i, \beta_j \ge 0$ not all zero (their sum is 1). □

Theorem 9 (Normalized KKT Condition at Locally Weakly Efficient Solutions)**.** Consider a locally weakly efficient solution $\hat{x}$ of the optimization problem $(\mathcal{MP})$. Suppose that, at $\hat{x}$, the σ_r-invex Slater constraint qualification is fulfilled and that the parameters r_j are nonnegative for all $j \in J(\hat{x})$. Under these assumptions, one can find coefficients $\mu_i \ge 0$, $i \in I$, normalized by $\sum_{i=1}^{p}\mu_i = 1$, as well as nonnegative multipliers η_j associated with the active constraints $j \in J(\hat{x})$, such that

$$0_n \in \sum_{i=1}^{p}\mu_i\,\partial_c\phi_i(\hat{x}) + \sum_{j\in J(\hat{x})}\eta_j\,\partial_c\psi_j(\hat{x}). \tag{7}$$

Proof. Let $\alpha_i \ge 0$ and $\beta_j \ge 0$ be the multipliers given by Theorem 8, for which (4) and (5) hold. In particular, there exist $\xi_i \in \partial_c\phi_i(\hat{x})$ and $\zeta_j \in \partial_c\psi_j(\hat{x})$ such that

$$0_n \in \partial_c\Theta(\hat{x}) = \sum_{i\in I}\alpha_i\xi_i + \sum_{j\in J(\hat{x})}\beta_j\zeta_j.$$

We first show that

$$\sum_{i\in I}\alpha_i > 0.$$

Suppose, to the contrary, that $\alpha_i = 0$ for all $i \in I$. Then

$$\sum_{j\in J(\hat{x})}\beta_j\zeta_j = 0_n, \qquad \sum_{j\in J(\hat{x})}\beta_j = 1.$$

By Proposition 1, there exists a Clarke-MFCQ direction d such that

$$\psi_j^0(\hat{x}; d) < 0, \qquad j \in J(\hat{x}).$$

Moreover, by the definition of the Clarke directional derivative,

$$\langle \zeta_j, d \rangle \leq \psi_j^0(\hat{x}; d) < 0, \qquad j \in J(\hat{x}).$$

Taking the inner product of the equality $\sum_{j \in J(\hat{x})} \beta_j \zeta_j = 0_n$ with d yields

$$0 = \left\langle \sum_{j \in J(\hat{x})} \beta_j \zeta_j, d \right\rangle = \sum_{j \in J(\hat{x})} \beta_j \langle \zeta_j, d \rangle \leq \sum_{j \in J(\hat{x})} \beta_j \psi_j^0(\hat{x}; d) < 0,$$

where the last strict inequality follows from $\sum_{j \in J(\hat{x})} \beta_j = 1$ and $\psi_j^0(\hat{x}; d) < 0$ for all $j \in J(\hat{x})$. This contradiction proves that

$$\sum_{i \in I} \alpha_i > 0.$$

We can therefore normalize the multipliers by defining

$$\mu_i := \frac{\alpha_i}{\sum\limits_{k \in I} \alpha_k}, \qquad i \in I, \qquad \eta_j := \frac{\beta_j}{\sum\limits_{k \in I} \alpha_k}, \qquad j \in J(\hat{x}).$$

Dividing (5) by $\sum_{k \in I} \alpha_k$ and using the above definitions gives (7). Furthermore,

$$\sum_{i \in I} \mu_i = \frac{\sum_{i \in I} \alpha_i}{\sum_{k \in I} \alpha_k} = 1.$$

Thus, the required normalized multiplier conditions are satisfied. □

As an immediate consequence of Theorem 9, we obtain the following corollary for scalar optimization problems, i.e., when $p = 1$ in problem $(\mathcal{MP})$.

Corollary 1. Assume that $\hat{x}$ is a local minimizer of the scalar optimization problem

$$\min \varphi(x), \quad \text{s.t.} \quad x \in \mathcal{S},$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a given locally Lipschitz function and $\mathcal{S}$ is defined as (2). If the σ_r-invex-SCQ is fulfilled at $\hat{x}$ and $r_j \geq 0$, for $j \in J(\hat{x})$, we can find some nonnegative multipliers η_j, for $j \in J(\hat{x})$, such that

$$0_n \in \partial_c \varphi(\hat{x}) + \sum_{j \in J(\hat{x})} \eta_j \, \partial_c \psi_j(\hat{x}). \tag{8}$$

We illustrate Theorem 9 on Example 1. At $\hat{x} = 0$, the vector 0_n is written, with $\mu_1 = \mu_2 = 1/2$ and $\eta_1 = 3/2$, as

$$0 = \mu_1 \xi_1 + \mu_2 \xi_2 + \eta_1 \cdot 1, \qquad \xi_1 = -1 \in \partial_c \phi_1(0) = [-1, 1], \;\; \xi_2 = -2 \in \partial_c \phi_2(0),$$

since $\frac{1}{2}(-1) + \frac{1}{2}(-2) + \frac{3}{2}(1) = -\frac{3}{2} + \frac{3}{2} = 0$; this confirms (7) at $\hat{x} = 0$.

Theorem 10 (Corollary: KKT Condition at Locally Efficient Solutions)**.** Consider a locally efficient solution $\hat{x}$ of the optimization problem $(\mathcal{MP})$. Suppose that, at $\hat{x}$, the σ_r-invex Slater constraint qualification is fulfilled and that the parameters r_j are nonnegative for all $j \in J(\hat{x})$. Under these assumptions, one can find coefficients $\mu_i \geq 0$, $i \in I$, normalized by $\sum_{i=1}^{p} \mu_i = 1$, as well as nonnegative multipliers η_j associated with the active constraints $j \in J(\hat{x})$, such that

$$0_n \in \sum_{i=1}^{p} \mu_i \partial_c \phi_i(\hat{x}) + \sum_{j \in J(\hat{x})} \eta_j \partial_c \psi_j(\hat{x}).$$

Proof. Since every locally efficient solution of $(\mathcal{MP})$ is also a locally weakly efficient solution, the result follows immediately from Theorem 9. □

It should be emphasized that the KKT multipliers μ_k corresponding to the objective functions are not necessarily positive in Theorems 9 and 10. In particular, some of these multipliers may be equal to zero for certain indices $k \in I$, which means that the associated objective functions ϕ_k do not influence the inclusion (7) in the characterization of weakly efficient or efficient solutions $\hat{x}$.

The result presented next shows that this phenomenon does not occur at a positive weighted-sum solution, in which case every component of the objective function actively contributes to the inclusion (7). Motivated by the potential degeneracy caused by zero-valued multipliers, several authors have introduced strengthened forms of KKT-type necessary conditions in the literature. In this context, the *strong KKT condition* for problem $(\mathcal{MP})$ is said to hold whenever the multipliers associated with the objective functions satisfy

$$\mu_i > 0 \quad \text{for all } i \in I.$$

Theorem 11 (KKT Condition for Positive Weighted-Sum Solutions)**.** Let $\hat{x}$ be a positive weighted-sum solution of the optimization problem $(\mathcal{MP})$, i.e., $\hat{x}$ locally minimizes $\sum_{i=1}^{p} \lambda_i \phi_i$ over $\mathcal{S}$ for some weights $\lambda_i > 0$, $i \in I$. Suppose that, at $\hat{x}$, the σ_r-invex Slater constraint qualification is fulfilled and that the parameters r_j are nonnegative for all $j \in J(\hat{x})$. Under these assumptions, one can find positive coefficients $\mu_i > 0$, $i \in I$, normalized by $\sum_{i=1}^{p} \mu_i = 1$, as well as nonnegative multipliers η_j associated with the active constraints $j \in J(\hat{x})$, such that

$$0_n \in \sum_{i=1}^{p} \mu_i \partial_c \phi_i(\hat{x}) + \sum_{j \in J(\hat{x})} \eta_j \partial_c \psi_j(\hat{x}).$$

Proof. By hypothesis $\hat{x}$ locally minimizes $\varphi(x) := \sum_{i=1}^{p} \lambda_i \phi_i(x)$ over $\mathcal{S}$ for some $\lambda_i > 0$, $i \in I$, so Corollary 1 yields

$$0_n \in \partial_c \varphi(\hat{x}) + \sum_{j \in J(\hat{x})} \hat{\eta}_j \partial_c \psi_j(\hat{x}),$$

for some $\hat{\eta}_j \geq 0$ as $j \in J(\hat{x})$. On the other hand, Theorem 1 yields

$$\partial_c \varphi(\hat{x}) = \partial_c \Big(\sum_{i=1}^{p} \lambda_i \phi_i \Big)(\hat{x}) \subseteq \sum_{i=1}^{p} \lambda_i \partial_c \phi_i(\hat{x}).$$

Consequently,

$$0_n \in \sum_{i=1}^{p} \lambda_i \partial_c \phi_i(\hat{x}) + \sum_{j \in J(\hat{x})} \hat{\eta}_j \partial_c \psi_j(\hat{x}).$$

Set $\tau := \sum_{i=1}^{p} \lambda_i > 0$. Taking $\mu_i := \dfrac{\lambda_i}{\tau}$, for $i \in I$, and $\eta_j := \dfrac{\hat{\eta}_j}{\tau}$, for $j \in J(\hat{x})$, we get the result. □

Remark 3. Note that the Fritz–John necessary condition (Theorem 8) holds at a locally weakly efficient solution $\hat{x}$ of $(\mathcal{MP})$ without requiring any constraint qualification. Specifically, it states that

$$0_n \in conv\big(\Phi_{\hat{x}} \cup \Psi_{\hat{x}}\big),$$

which gives multipliers $\mu_i \geq 0$ and $\eta_j \geq 0$, for $i \in I$ and $j \in J(\hat{x})$, not all zero, satisfying (7). However, in this framework it is possible that $\mu_i = 0$ for all $i \in I$, meaning that the objective functions may not contribute to the inclusion (7). To overcome this limitation, we introduced the σ_r-Invex-SCQ, which by Theorem 9 yields the normalized KKT necessary condition

$$0_n \in conv\big(\Phi_{\hat{x}}\big) + cone\big(\Psi_{\hat{x}}\big),$$

ensuring that at least one multiplier μ_i is strictly positive. Furthermore, under the same σ_r-Invex-SCQ assumption, Theorem 11 establishes the strong KKT condition at positive weighted-sum solutions. Note that if the objective functions ϕ_i, for $i \in I$, are continuously differentiable at $\hat{x}$, [24, Theorem 6.9] implies that the strong KKT condition can be rewritten as

$$0_n \in ri\big(conv(\Phi_{\hat{x}})\big) + cone(\Psi_{\hat{x}}),$$

where $ri(A)$ denotes the relative interior of a convex set $A \subseteq \mathbb{R}^n$; see [12, 29] for details.

For locally isolated efficient solutions, we use the term *perturbed-KKT condition*: there is $\nu > 0$ such that

$$\nu \mathbb{B}_n \subseteq conv(\Phi_{\hat{x}}) + cone(\Psi_{\hat{x}}),$$

equivalently, $0_n \in int\big(conv(\Phi_{\hat{x}}) + cone(\Psi_{\hat{x}})\big)$. This terminology is chosen deliberately: analogous inclusions already appear in the isolated-efficiency literature for the convex semi-infinite case [8] and for related nonsmooth settings [27]; the theorem below derives it here for finitely many locally Lipschitz constraints from the active invex-Slater sufficient condition for Clarke-MFCQ.

We first isolate, as a standalone lemma, the Clarke normal-cone upper estimate under Clarke-MFCQ. Unlike the argument used in an earlier draft of this paper, this estimate requires no regularity assumption on $\mathcal{S}$; the underlying mechanism is the standard hypertangent construction for Clarke tangent cones.

Lemma 1 (Normal-cone upper estimate under Clarke-MFCQ)**.** The feasible set $\mathcal{S}$ in (2) is closed. If Clarke-MFCQ holds at $\hat{x} \in \mathcal{S}$, then

$$N(\mathcal{S}, \hat{x}) \subseteq \sum_{j \in J(\hat{x})} \mathbb{R}_+ \partial_c \psi_j(\hat{x}) = cone\Big(\bigcup_{j \in J(\hat{x})} \partial_c \psi_j(\hat{x}) \Big) = cone(\Psi_{\hat{x}}).$$

Proof. Put $K := cone(\Psi_{\hat{x}})$ and $L := \{v \in \mathbb{R}^n \mid \psi_j^0(\hat{x}; v) \leq 0 \, \forall j \in J(\hat{x})\}$. Since $\psi_j^0(\hat{x}; v) = \max_{\xi \in \partial_c \psi_j(\hat{x})} \langle \xi, v \rangle$ (Theorem 1(iii)), we have $L = K^{\leq}$. Let d be a Clarke-MFCQ direction, so $\psi_j^0(\hat{x}; d) < 0$ for every $j \in J(\hat{x})$. For $v \in L$ and $\varepsilon > 0$, subadditivity and positive homogeneity of $\psi_j^0(\hat{x}; \cdot)$ give

$$\psi_j^0(\hat{x}; v + \varepsilon d) \leq \psi_j^0(\hat{x}; v) + \varepsilon\, \psi_j^0(\hat{x}; d) < 0, \qquad j \in J(\hat{x}).$$

A strict Clarke descent direction for every active constraint is a hypertangent direction to $\mathcal{S}$: by the definition of the Clarke directional derivative, for each active j the strict inequality $\psi_j^0(\hat{x}; v + \varepsilon d) < 0$ persists uniformly over nearby feasible base points, nearby directions, and sufficiently small positive step sizes, while inactive constraints remain satisfied by continuity of ψ_j, $j \notin J(\hat{x})$. This is the standard hypertangent mechanism for Clarke tangent cones; see [4, Chapter 2] and [2, Definition 3.3]. Hence $v + \varepsilon d \in \Gamma(\mathcal{S}, \hat{x})$ for all $\varepsilon > 0$; letting $\varepsilon \downarrow 0$ and using closedness of $\Gamma(\mathcal{S}, \hat{x})$ gives $L \subseteq \Gamma(\mathcal{S}, \hat{x})$. Taking polars reverses inclusions, so

$$N(\mathcal{S}, \hat{x}) = \Gamma(\mathcal{S}, \hat{x})^{\leq} \subseteq L^{\leq} = K.$$

Finally, $\Psi_{\hat{x}}$ is compact, and d separates $\Psi_{\hat{x}}$ strictly from the origin ($\langle \xi, d \rangle \leq \psi_j^0(\hat{x}; d) < 0$ for $\xi \in \partial_c \psi_j(\hat{x})$, $j \in J(\hat{x})$), so $0_n \notin conv(\Psi_{\hat{x}})$; by Theorem 3, $K = cone(\Psi_{\hat{x}})$ is closed, completing the proof. If $J(\hat{x}) = \emptyset$, the estimate is immediate with $cone(\emptyset) := \{0_n\}$. $\square$

Theorem 12 (Perturbed-KKT Necessary Condition at Locally Isolated Efficient Solutions)**.** Let $\hat{x}$ be a locally isolated efficient solution of the optimization problem $(\mathcal{MP})$ with isolation constant $\nu > 0$. Suppose that, at $\hat{x}$, the σ_r-invex Slater constraint qualification is fulfilled and that the parameters r_j are nonnegative for all $j \in J(\hat{x})$. Then

$$\nu \mathbb{B}_n \subseteq conv(\Phi_{\hat{x}}) + cone(\Psi_{\hat{x}}).$$

Proof. Set $G(x) := \max_{1 \leq i \leq p} \{\phi_i(x) - \phi_i(\hat{x})\}$. Local isolated efficiency means that $\hat{x}$ locally minimizes $G(x) - \nu \|x - \hat{x}\|$ over $\mathcal{S}$. Let $v \in \partial\big(\nu \| \cdot - \hat{x}\|\big)(\hat{x})$ be arbitrary; by definition of the convex subdifferential (Equation (1)), $\nu\|x - \hat{x}\| - \nu\|\hat{x} - \hat{x}\| \geq \langle v, x - \hat{x} \rangle$ for all x, so $\hat{x}$ also locally minimizes $G(x) - \langle v, x \rangle$ over $\mathcal{S}$. By Theorem 7 applied to $f(\cdot) = G(\cdot) - \langle v, \cdot \rangle$,

$$0_n \in \partial_c\big(G(\cdot) - \langle v, \cdot\rangle\big)(\hat{x}) + N(\mathcal{S}, \hat{x}) \subseteq \partial_c G(\hat{x}) - v + N(\mathcal{S}, \hat{x}),$$

i.e., $v \in \partial_c G(\hat{x}) + N(\mathcal{S}, \hat{x})$. Since $v \in \partial\big(\nu\|\cdot - \hat{x}\|\big)(\hat{x})$ was arbitrary and $\partial\big(\nu\|\cdot - \hat{x}\|\big)(\hat{x}) = \nu\mathbb{B}_n$ (see, e.g., [12, Section VI.4]),

$$\nu\mathbb{B}_n \subseteq \partial_c G(\hat{x}) + N(\mathcal{S}, \hat{x}).$$

By Theorem 2, $\partial_c G(\hat{x}) \subseteq conv(\Phi_{\hat{x}})$. By Proposition 1, the active (σ, r)-Slater condition at $\hat{x}$ implies Clarke-MFCQ, so Lemma 1 gives $N(\mathcal{S}, \hat{x}) \subseteq cone(\Psi_{\hat{x}})$. Combining these three inclusions yields

$$\nu\mathbb{B}_n \subseteq conv(\Phi_{\hat{x}}) + cone(\Psi_{\hat{x}}),$$

which completes the proof. □

The next result is an immediate multiplier-form reformulation of Theorem 12, obtained directly from the convex-hull and conic-hull representations of Theorem 3; it is recorded separately because it is the form used to interpret the perturbed-KKT inclusion pointwise.

Corollary 2 (Multiplier Form of the Perturbed-KKT Condition)**.** Under the hypotheses of Theorem 12, for every $w \in \nu\mathbb{B}_n$ there exist $\mu_i \geq 0, i \in I$, with $\sum_{i=1}^{p} \mu_i = 1$, and $\eta_j \geq 0, j \in J(\hat{x})$, such that

$$w \in \sum_{i=1}^{p} \mu_i \partial_c \phi_i(\hat{x}) + \sum_{j \in J(\hat{x})} \eta_j \partial_c \psi_j(\hat{x}). \tag{9}$$

Proof. By Theorem 12, $w \in conv(\Phi_{\hat{x}}) + cone(\Psi_{\hat{x}})$ for every $w \in \nu\mathbb{B}_n$; Theorem 3 applied to $\Phi_{\hat{x}} = \bigcup_{i \in I} \partial_c \phi_i(\hat{x})$ and $\Psi_{\hat{x}} = \bigcup_{j \in J(\hat{x})} \partial_c \psi_j(\hat{x})$ gives the stated multiplier representation (9). □

We illustrate Theorem 12 and Corollary 2 on Example 1, where $\hat{x} = 0$ is globally isolated efficient with constant $\nu = 2$, $\partial_c\phi_1(0) = [-1, 1]$, $\partial_c\phi_2(0) = \{-2\}$, and $\partial_c\psi(0) = \{1\}$:

$$conv(\Phi_0) = [-2, 1], \qquad cone(\Psi_0) = [0, \infty), \qquad conv(\Phi_0) + cone(\Psi_0) = [-2, \infty),$$

and hence $2\mathbb{B}_1 = [-2, 2] \subseteq [-2, \infty)$, confirming the perturbed-KKT inclusion at $\hat{x} = 0$.

It is straightforward, using the strict inequality in Definition 4, to construct examples that fail the classical SCQ (Definition 3) yet satisfy σ_r-Invex-SCQ at the point of interest; Example 1 is one such instance, since ψ is nonconvex there. The recent literature on constraint qualifications for nonsmooth (multiobjective, semi-infinite, or manifold-based) programming [8, 10, 11, 14, 15, 19, 20, 27, 28, 30, 34, 35] develops sufficient conditions or works in settings (semi-infinite index sets, convexificators, Riemannian or Hadamard manifolds) that are structurally different from the finite, Euclidean, active-set setting considered here, so the present results can be viewed as complementary to, rather than subsumed by, that literature.

4 Conclusions

We studied a finite nonsmooth multiobjective problem under an active-set Slater test expressed through pointwise $(\sigma_{\hat{x}}, r_j)$-invex inequalities. For every map selected through a structurally restricted admissible class or selection rule $\Sigma(\psi, \hat{x})$ prescribed before the qualification is tested and not constructed from an MFCQ direction, the test implies Clarke-MFCQ (Proposition 1) and therefore yields the usual normalized Clarke–KKT and normal-cone consequences (Theorems 9, 10, and 11). If $\Sigma(\psi, \hat{x})$ contains every locally Lipschitz map, Clarke-MFCQ is equivalent to the class-level condition with $r = 0$. The present proofs also use the global invex inequality only at one active Slater point, so a one-point certificate together with the resulting Clarke-MFCQ direction is the essential ingredient throughout. For locally isolated efficient solutions, the same mechanism, combined with a Clarke normal-cone upper estimate that requires no regularity assumption on the feasible set (Lemma 1), gives the perturbed-KKT type inclusion of Theorem 12 and Corollary 2. Taken together, these results show that the value of the active (σ, r)-Slater condition lies not in weakening Clarke-MFCQ — it is, after all, only a sufficient condition for it — but in supplying, at a single test point and without any convexity assumption, an explicit constructive certificate for Clarke-MFCQ whenever the admissible class $\Sigma(\psi, \hat{x})$ is genuinely restricted by the problem's structure. A natural direction for future work is to identify problem classes (e.g., transformations of convex or genuinely invex data) for which such a nontrivial restriction of $\Sigma(\psi, \hat{x})$ arises naturally, together with a converse statement quantifying how much weaker than Clarke-MFCQ the resulting class-level condition can be.

Declarations

Availability of Supporting Data

The data that support the findings of this study are available from the corresponding author upon reasonable request. Due to organizational confidentiality, the data are not publicly available.

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The authors declare no known competing financial interests or personal relationships that could have influenced the work reported in this paper.

Authors Contributions

Parastoo Heiatian Naeini: Conceptualization; Methodology; Software and Validation; Formal Analysis; Investigation; Resources; Data Curation; Writing – Original Draft; Writing – Review & Editing; Visualization; Supervision; Project Administration. **Ahmad Rezaee**: Conceptualization; Software and Validation; Formal Analysis; Resources; Data Curation; Writing – Original Draft.

Artificial Intelligence Statement

Artificial intelligence (AI) tools, including large language models, were used solely for language editing and improving readability. AI tools were not used for generating ideas, performing analyses, interpreting results, or writing the scientific content. All scientific conclusions and intellectual contributions were made exclusively by the authors.

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