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Balanced Pythagorean Fuzzy Superhypergraphs with an Illustrative Application to Hypernetwork Modeling

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Abstract. Complex hypernetworks are important in real-world problems, and their regularity and balance are particularly significant; hence, suitable mathematical tools are needed. In this regard, this paper introduces the notion of refined Pythagorean fuzzy superhypergraphs (RPFSs) on an arbitrary non-empty set, for the clustering of complex hypernetworks. We introduce the complement of an RPFS, classify RPFSs as complementary, self-complementary, complete, and regular, and investigate their fundamental properties. The notion of isomorphic RPFSs is established, which leads to self-complementary RPFSs, and conditions under which an RPFS is self-complementary are considered. For any RPFS, we introduce density measures for Pythagorean fuzzy links and Pythagorean fuzzy superedges, which serve as measures of coherence for hypernetworks, and we analyze these measures for constant complete and self-complementary RPFSs. The notions of Pythagorean fuzzy link-degree and superedge-degree are established, yielding positive regular hyperstructures whose densities are computed. Finally, balanced RPFSs are introduced, characterized via density, and analyzed under isomorphism.

Keywords. k-density, (r,s)-regular RPFS, self-complementary RPFS, (c,c')-constant RPFS, balanced RPFS.

MSC. 03E72; 05C72; 05C65.

1 Introduction

A fuzzy graph, as a generalization of a graph, is a mathematical tool for modeling real data in complex networks [22]. Fuzzy graphs now have many real-world applications, and numerous researchers work in this area [16, 17, 18, 19, 23]. A superhypergraph is a tool for connecting real data in a clustered and grouped form and is therefore of great importance today. This concept generalizes graphs by connecting more than two elements simultaneously, and it is mainly used for problems that involve group connections. Notable contributions in this direction include the studies of Hamidi et al. on the spectrum of superhypergraphs, on valued fuzzy superhypergraphs, and on the domination number of superhypergraphs [12, 13, 14]. In complex hypernetworks, connectivity is not the only important criterion: the type of a connection and its value and size also have a strong effect. Consequently, various kinds of fuzzy subsets have been introduced, including Pythagorean fuzzy subsets (PFSs). PFSs, introduced by Yager [25], have successfully addressed imprecision in graph theory, and they now have applications in many branches of mathematics, as in [1, 2, 3, 4, 7, 9, 10, 24].

Related Work. The density of a graph reflects the pattern of connections in the whole network. As networks grow rapidly, problems on them become increasingly uncertain, so balancing a network is an important issue. A graph is called balanced if the density of each of its subgraphs does not exceed the density of the graph itself. In the fuzzy setting, Al-Hawary et al. introduced balanced fuzzy graphs and studied some operations on fuzzy graphs [5, 6]. As a generalization of balanced fuzzy graphs, Karunambigai et al. introduced balanced intuitionistic fuzzy graphs and presented some of their properties [15]. Picture fuzzy graphs are an extension of intuitionistic fuzzy graphs, and the balanced picture fuzzy graph, a special type of picture fuzzy graph, was introduced by Amanathulla et al. [8].

Motivation. In today's world, collective communication plays a major role in solving the problems we face. Therefore, a mathematical tool that can model such problems and solve them on the basis of the resulting model is of great importance. Our motivation is to create a mathematical model that can examine collective communication, individual communication, and the communication of a component within the whole set, using the information of a given problem. Although fuzzy graphs are useful for designing complex networks, they can examine the effect of only two elements at a time and cannot capture the overlap of more than two elements simultaneously. Fuzzy superhypergraphs were therefore proposed to address this need. Graph density is important because it quantifies the overall connectivity and coherence of a network and distinguishes sparse structures from highly interconnected ones in real-world systems. In fuzzy graphs, density extends classical connectivity by including the membership values of vertices and edges, which provides a robust measure of overall structural coherence under uncertainty. Its primary importance lies in defining balanced fuzzy graphs, in which no subgraph is disproportionately dense. Considering the importance of superhypergraphs and Pythagorean fuzzy concepts, in this study we aim to diagnose properly the coherence or dispersion of grouped and clustered data in order to support optimal decision making. To this end, we extend the notions of balanced fuzzy, intuitionistic fuzzy, and picture fuzzy graphs, which are used for complex networks, to balanced refined Pythagorean fuzzy superhypergraphs (RPFSs), and we apply them to complex hypernetworks.

Contributions and Organization. We combine the concepts of superhypergraphs and Pythagorean fuzzy subsets to design a purposeful complex hypernetwork. The notion of an RPFS is introduced on an arbitrary non-empty set, together with the complement of an RPFS. Complete, (c, c')-constant, and regular RPFSs are introduced and investigated. The concept of isomorphic RPFSs is presented, and self-complementary RPFSs are analyzed. For every $k \in \mathbb{N}$ and every RPFS, we introduce the k-density of Pythagorean fuzzy links, of Pythagorean fuzzy superedges, and of the RPFS itself. We investigate the k-density of (c, c')-constant complete RPFSs and of self-complementary RPFSs. The Pythagorean fuzzy link-degree and superedge-degree of a vertex are introduced, and the concept of an (r, s)-regular RPFS is defined for $r, s \in \mathbb{R}$. We then compute the k-density of (r, s)-regular RPFSs and investigate its properties. Finally, we consider an arbitrary RPFS, introduce its Pythagorean fuzzy subsuperhypergraphs, and study the relation between the subsuperhypergraphs of isomorphic RPFSs. The notion of a balanced RPFS is presented via the density of its Pythagorean fuzzy subsuperhypergraphs, isomorphic balanced RPFSs are analyzed, and a class of balanced RPFSs is introduced.

The remainder of this paper is organized as follows. Section 2 recalls the basic definitions. Section 3 introduces RPFSs and studies their complements, complete and constant RPFSs, and isomorphism. Section 4 defines the k-density and the regularity of RPFSs, and introduces balanced RPFSs. Section 5 concludes the paper.

2 Preliminaries

Throughout the paper, X denotes a non-empty set, $\mathrm{FS}(X)$ denotes the set of all fuzzy subsets of X, and $\mathrm{PFS}(X)$ denotes the set of all Pythagorean fuzzy subsets (PFSs) of X. This section recalls the basic definitions used in the sequel.

Definition 1. A fuzzy subset of X [20] is a membership function $\mu\colon X \to [0, 1]$.

Definition 2. A Pythagorean fuzzy subset (PFS) of X [25] is an ordered pair $P = (\mu_P, \nu_P)$ of fuzzy subsets of X satisfying the Pythagorean inequality $\mu_P(x)^2 + \nu_P(x)^2 \leq 1$ for all $x \in X$.

Definition 3. A Pythagorean fuzzy graph (PFG) on X [21] is an ordered pair of PFSs that satisfy the fuzzy graph and anti-fuzzy graph relations.

Definition 4. Let $X \neq \emptyset$ [12].

(i) A quasi-superhypergraph (QSH) on X is a system of supervertices S_i and links $\varphi_{i,j} : S_i \to S_j$ such that the supervertices cover X, that is, $\bigcup_i S_i = X$.

(ii) A QSH on X is called a superhypergraph (SH) if every supervertex is linked to another supervertex.

Definition 5. For a PFS $P = (\mu_P, \nu_P)$ on X, the *support* of P is

$$P^* = \{x \in X \mid \mu_P(x) > 0,\ \nu_P(x) < 1\}.$$

The support of a family of PFSs is the family of the supports of its members.

Definition 6. Let $\mathcal{V} = \{P_i = (\mu_{P_i}, \nu_{P_i})\}_{i=1}^{v}$, $\mathcal{E} = \{Q_{ij} = (\mu_{Q_{ij}}, \nu_{Q_{ij}})\}_{i,j\geq 1}^{e}$, and $\mathcal{D} = \{R_i = (\mu_{R_i}, \nu_{R_i})\}_{i=1}^{d}$ be finite families of non-trivial PFSs of X. Then $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ is a *Pythagorean fuzzy quasi-superhypergraph* (PFQS) on X [11] if $X = \bigcup_{i=1}^{v} P_i^*$ and

(i) for all $xy \in R_i^*$,

$$\mu_{R_i}(xy) \leq \min\{\mu_{P_i}(x), \mu_{P_i}(y)\},$$
$$\nu_{R_i}(xy) \geq \max\{\nu_{P_i}(x), \nu_{P_i}(y)\};$$

(ii) for all $xy \in Q_{ij}^*$,

$$\mu_{Q_{ij}}(xy) \leq \min\Big\{\frac{\mu_{P_i}(x)}{w(P_i)}, \frac{\mu_{P_j}(y)}{w(P_j)}\Big\},$$
$$\nu_{Q_{ij}}(xy) \geq \max\Big\{\frac{\nu_{P_i}(x)}{w(P_i)}, \frac{\nu_{P_j}(y)}{w(P_j)}\Big\},$$

where $w(P_i) = \sum_{x\in P_i^*}\big(\mu_{P_i}(x) + \nu_{P_i}(x)\big)$. We call $\mathcal{V}$ the Pythagorean fuzzy supervertices, $\mathcal{E}$ the Pythagorean fuzzy superedges, and $\mathcal{D}$ the Pythagorean fuzzy links. A PFQS $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ on X is called a *Pythagorean fuzzy superhypergraph* (PFSh) on X if $|\mathcal{V}| \leq |\mathcal{E}|$ and $|R_i^*| \geq 1$ for every $1 \leq i \leq d$. A PFSh is called *soft* if $w(P_i) < 1$ for every $1 \leq i \leq v$, *rough* if $w(P_i) > 1$ for every $1 \leq i \leq v$, and *smooth* if $w(P_i) = 1$ for every $1 \leq i \leq v$. A PFQS is called *coverable* if every member of $\mathcal{E}$ is an onto map and

$$\mu_{R_i}(xy) + \mu_{P_i}(x) \leq \nu_{R_i}(xy) + \nu_{P_i}(x),$$
$$\mu_{R_i}(xy) + \mu_{P_i}(y) \leq \nu_{R_i}(xy) + \nu_{P_i}(y),$$

and similarly for Q_{ij}.

3 Refined Pythagorean Fuzzy Superhypergraphs

In this section, we introduce the notion of a refined Pythagorean fuzzy superhypergraph (RPFS) on an arbitrary non-empty set and the complement of an RPFS. Complete and (c, c')-constant RPFSs are introduced, and their properties are investigated. The concept of isomorphic RPFSs is also presented, and self-complementary RPFSs are analyzed. From now on, X is a non-empty set with $|X| \geq 1$, and supports are understood in the sense of Definition 5.

Definition 7. Let $\mathcal{V} = \{P_i = (\mu_{P_i}, \nu_{P_i})\}_{i=1}^{v}$, $\mathcal{E} = \{Q_{ij} = (\mu_{Q_{ij}}, \nu_{Q_{ij}})\}_{i,j\geq 1}^{e}$, and $\mathcal{D} = \{R_i = (\mu_{R_i}, \nu_{R_i})\}_{i=1}^{d}$ be finite families of non-trivial PFSs of X. Then $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ is a *refined Pythagorean fuzzy superhypergraph* (RPFS) on X if $X = \bigcup_{i=1}^{v} P_i^*$, $P_i^* \cap P_j^* = \emptyset$ for all $i \neq j$, and

(i) for all $xy \in R_i^*$,

$$\mu_{R_i}(xy) \leq \min\{\mu_{P_i}(x), \mu_{P_i}(y)\},$$
$$\nu_{R_i}(xy) \geq \max\{\nu_{P_i}(x), \nu_{P_i}(y)\};$$

(ii) for all $xy \in Q_{ij}^*$,

$$\mu_{Q_{ij}}(xy) \leq \min\{\mu_{P_i}(x), \mu_{P_j}(y)\},$$
$$\nu_{Q_{ij}}(xy) \geq \max\{\nu_{P_i}(x), \nu_{P_j}(y)\}.$$

For $1 \leq i \leq v$ with $|P_i^*| \geq 2$, we require $|P_i^*| - 1 \leq |R_i^*| \leq \binom{|P_i^*|}{2}$. In addition, for every $1 \leq i \leq v$ there must exist some $1 \leq j \leq v$ such that $Q_{ij}^* \subseteq P_i^* \times P_j^*$ or $Q_{ji}^* \subseteq P_j^* \times P_i^*$. If $|\mathcal{V}| = 1$, the RPFS is called *singleton*. We call $\mathcal{V}$ the Pythagorean fuzzy supervertices, $\mathcal{E}$ the Pythagorean fuzzy superedges, and $\mathcal{D}$ the Pythagorean fuzzy links, and we write $H^* = (\mathcal{V}^*, \mathcal{E}^*, \mathcal{D}^*)$ for the support of H.

From now on, we consider only non-singleton RPFSs.

Example 1. Let $X = \{p_i\}_{i=1}^{9}$ be the students of a school, who belong to three classes P_1, P_2, and P_3. The students want to carry out a group project for the academic progress of their school. Each student p_i is identified by two characteristics, intelligence and perseverance, and is written $(p_i, \text{intelligence}, \text{perseverance})$. The relation between two students p_i and p_j of the same class is identified by two characteristics, understanding and cooperation, and is written $(p_ip_j, \text{understanding}, \text{cooperation})$. A student p_i of one class can be linked to a student p_j of another class by two characteristics, interaction and competition, written $(p_ip_j, \text{interaction}, \text{competition})$. The collected information is represented by the RPFS H shown in Figure 1.

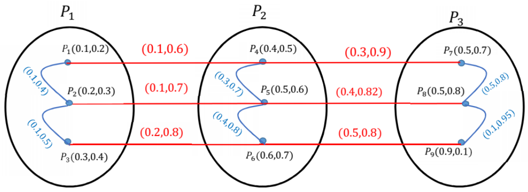

Figure 1: A complex hypernetwork model of the school in Example 1.

Definition 8. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be an RPFS on X. Then H is called *complete* if

(i) for all $x, y \in P_i^*$,

$$\mu_{R_i}(xy) = \min\{\mu_{P_i}(x), \mu_{P_i}(y)\},$$
$$\nu_{R_i}(xy) = \max\{\nu_{P_i}(x), \nu_{P_i}(y)\};$$

(ii) for all $x \in P_i^*$ and $y \in P_j^*$,

$$\mu_{Q_{ij}}(xy) = \min\{\mu_{P_i}(x), \mu_{P_j}(y)\},$$
$$\nu_{Q_{ij}}(xy) = \max\{\nu_{P_i}(x), \nu_{P_j}(y)\}.$$

A complete RPFS is denoted by $K_{|X|}$. Indeed, items (i) and (ii) imply that

$$\left|\bigcup_{1\leq i,j\leq e} Q_{ij}^*\right| = \binom{|X|}{2} - \left|\bigcup_i R_i^*\right|.$$

Example 2. Let $X = \{p_i\}_{i=1}^6$. Then H is a complete RPFS on X, as shown in Figure 2.

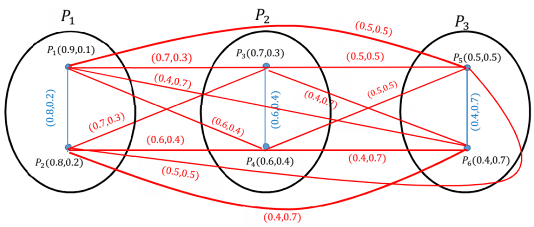

Figure 2: A complete RPFS on $X = \{p_i\}_{i=1}^6$.

Definition 9. An RPFS $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ is called (c, c')-*constant* if

$$\mathcal{V} = \{P_i = (\mu_{P_i}, \nu_{P_i}) = (c, c')\}_{i=1}^v,$$
$$\mathcal{E} = \{Q_{ij} = (\mu_{Q_{ij}}, \nu_{Q_{ij}}) = (c, c')\}_{i,j\geq 1}^e,$$
$$\mathcal{D} = \{R_i = (\mu_{R_i}, \nu_{R_i}) = (c, c')\}_{i=1}^d,$$

where $c, c' \in \mathbb{R}^{>0}$ and $c^2 + c'^2 \leq 1$.

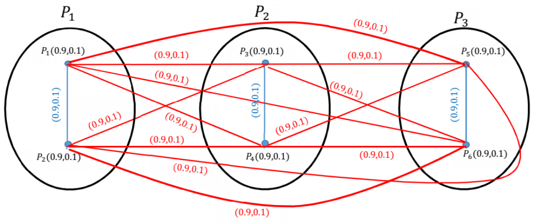

Figure 3: A (c, c')-constant complete RPFS on $X = \{p_i\}_{i=1}^{6}$.

Example 3. Let $X = \{p_i\}_{i=1}^{6}$. Then H is a complete RPFS on X that is (c, c')-constant, as shown in Figure 3.

Definition 10. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be an RPFS on X. The *complement* of H is $H^c = (\mathcal{V}^c, \mathcal{E}^c, \mathcal{D}^c)$, defined as follows:

(i) for $x \in P_i^*$, $\big(\mu_{P_i}^c(x), \nu_{P_i}^c(x)\big) = \big(1 - \nu_{P_i}(x), 1 - \mu_{P_i}(x)\big)$;

(ii) for $xy \in P_i^* \times P_i^*$, $\big(\mu_{R_i}^c(xy), \nu_{R_i}^c(xy)\big) = \big(1 - \nu_{R_i}(xy), 1 - \mu_{R_i}(xy)\big)$;

(iii) for $xy \in P_i^* \times P_j^*$, $\big(\mu_{Q_{ij}}^c(xy), \nu_{Q_{ij}}^c(xy)\big) = \big(1 - \nu_{Q_{ij}}(xy), 1 - \mu_{Q_{ij}}(xy)\big)$.

By Definition 10, we have the following lemma.

Lemma 1. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be an RPFS on X. Then $(P_i^c)^* = P_i^*$, $(R_i^c)^* = R_i^*$, and $(Q_{ij}^c)^* = Q_{ij}^*$.

Proof. For $x \in X$, we have $\mu_{P_i}^c(x) > 0$ and $\nu_{P_i}^c(x) < 1$ if and only if $1 - \nu_{P_i}(x) > 0$ and $1 - \mu_{P_i}(x) < 1$, that is, if and only if $\nu_{P_i}(x) < 1$ and $\mu_{P_i}(x) > 0$. The same argument applies to R_i and Q_{ij}. $\square$

Example 4. Consider the RPFS H in Figure 1, and let $p_a \in P_3^*$ be a student with

$$\big(\mu_{P_3}(p_a), \nu_{P_3}(p_a)\big) = (0.2, 0.2).$$

Then

$$\big(1 - \nu_{P_3}(p_a)\big)^2 + \big(1 - \mu_{P_3}(p_a)\big)^2 = 1.28 > 1,$$

so P_3^c is not a PFS. Thus, H^c is not an RPFS.

Example 4 shows that the complement of an RPFS on X is not necessarily an RPFS on X. For $J \subseteq [0, 1]^2$, we write $H|_J$ for an RPFS H with $\mathcal{V} = \{P_i = (\mu_{P_i}, \nu_{P_i})\}_{i=1}^{v}$ such that $\big(\mu_{P_i}(x), \nu_{P_i}(x)\big) \in J$ for every $x \in X$ and every $1 \le i \le v$.

Definition 11. An RPFS H on X is called *complementary* if there exists $J \subseteq [0, 1]^2$ such that $(H|_J)^c$ is an RPFS.

Theorem 1. Let $X \neq \emptyset$. Then there exists a subset $J \subseteq [0,1]^2$ such that, if $H|_J$ is an RPFS on X, then $(H|_J)^c$ is an RPFS.

Proof. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be an RPFS on X and let $xy \in R_i^*$. Since H is an RPFS, Definition 7 gives

$$\begin{aligned}
\mu^c_{R_i}(xy) &= 1 - \nu_{R_i}(xy) \leq 1 - \max\{\nu_{P_i}(x), \nu_{P_i}(y)\} \\
&= \min\{1 - \nu_{P_i}(x),\ 1 - \nu_{P_i}(y)\} = \min\{\mu^c_{P_i}(x), \mu^c_{P_i}(y)\}, \\
\nu^c_{R_i}(xy) &= 1 - \mu_{R_i}(xy) \geq 1 - \min\{\mu_{P_i}(x), \mu_{P_i}(y)\} \\
&= \max\{1 - \mu_{P_i}(x),\ 1 - \mu_{P_i}(y)\} = \max\{\nu^c_{P_i}(x), \nu^c_{P_i}(y)\}.
\end{aligned}$$

Similarly, for $xy \in Q_{ij}^*$ we obtain

$$\mu^c_{Q_{ij}}(xy) \leq \min\{\mu^c_{P_i}(x), \mu^c_{P_j}(y)\} \ \text{ and } \ \nu^c_{Q_{ij}}(xy) \geq \max\{\nu^c_{P_i}(x), \nu^c_{P_j}(y)\}.$$

By Definition 10, H^c is therefore an RPFS provided that $\mathcal{V}^c = \{P_i^c\}_{i=1}^v$, $\mathcal{E}^c = \{Q_{ij}^c\}_{i,j\geq 1}^e$, and $\mathcal{D}^c = \{R_i^c\}_{i=1}^d$ are finite families of non-trivial PFSs of X.

Let $x \in (P_i^c)^*$. Then $\mu^c_{P_i}(x)^2 + \nu^c_{P_i}(x)^2 \leq 1$ holds if and only if $\big(1 - \nu_{P_i}(x)\big)^2 + \big(1 - \mu_{P_i}(x)\big)^2 \leq 1$. Put $a_i = \mu_{P_i}(x)$ and $b_i = \nu_{P_i}(x)$, so $a_i, b_i \in [0,1]$. Since H is an RPFS, $a_i^2 + b_i^2 \leq 1$. Hence we must solve the system of inequalities

$$\begin{cases} a_i^2 + b_i^2 \leq 1, \\ (a_i - 1)^2 + (b_i - 1)^2 \leq 1. \end{cases}$$

Clearly, its solution set is the region J_1 shown in Figure 4. In the same way, $\mathcal{E}^c$ and $\mathcal{D}^c$ consist

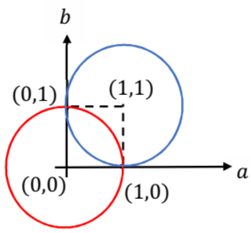

Figure 4: The region $J_1 = \left\{(\xi,\eta) \mid 1 - \sqrt{1-(\xi-1)^2} \leq \eta \leq \sqrt{1-\xi^2},\ \xi,\eta \in [0,1]\right\}$.

of PFSs of X if the membership pairs of the superedges and of the links lie in suitable regions J_2 and J_3, respectively, obtained by the same computation. Hence, there exists a subset $J = J_1 \cap J_2 \cap J_3 \subseteq [0,1]^2$ such that $(H|_J)^c$ is an RPFS. □

Corollary 1. Let $H|_J$ be an RPFS on X with $J = \{(a,b) \mid a + b = 1\}$. Then H^c is an RPFS on X (see Figure 5).

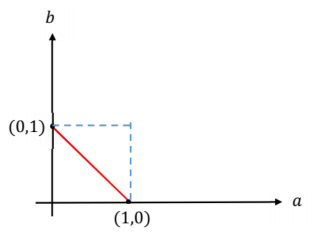

Figure 5: The region $J = \{(a, b) \mid b = 1 - a\}$.

Definition 12. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ and $H' = (\mathcal{V}', \mathcal{E}', \mathcal{D}')$ be RPFSs on X and X', respectively. Then H and H' are *isomorphic*, denoted $H \cong H'$, if there exists a bijection $\varphi \colon X \to X'$ such that

(i) for every $x \in P_i^*$, $\big(\mu_{P_i}(x), \nu_{P_i}(x)\big) = \big(\mu_{P_i'}(\varphi(x)), \nu_{P_i'}(\varphi(x))\big)$;

(ii) for every $xy \in P_i^* \times P_i^*$, $\big(\mu_{R_i}(xy), \nu_{R_i}(xy)\big) = \big(\mu_{R_i'}(\varphi(x)\varphi(y)), \nu_{R_i'}(\varphi(x)\varphi(y))\big)$;

(iii) for every $xy \in P_i^* \times P_j^*$, $\big(\mu_{Q_{ij}}(xy), \nu_{Q_{ij}}(xy)\big) = \big(\mu_{Q_{ij}'}(\varphi(x)\varphi(y)), \nu_{Q_{ij}'}(\varphi(x)\varphi(y))\big)$.

Definition 13. An RPFS $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ on X is called *self-complementary* if $H \cong H^c$.

Theorem 2. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be a self-complementary RPFS on X. Then

(i)

$$\sum_{i=1}^{|\mathcal{V}|} \sum_{x \in P_i^*} \mu_{P_i}(x) + \sum_{i=1}^{|\mathcal{V}|} \sum_{x \in P_i^*} \nu_{P_i}(x) = |X|,$$

(ii)

$$\sum_{i=1}^{|\mathcal{D}|} \sum_{xy \in P_i^* \times P_i^*} \mu_{R_i}(xy) + \sum_{i=1}^{|\mathcal{D}|} \sum_{xy \in P_i^* \times P_i^*} \nu_{R_i}(xy) = \sum_{i=1}^{|\mathcal{D}|} |P_i^* \times P_i^*|,$$

(iii)

$$\sum_{Q_{ij}\in\mathcal{E}}\sum_{xy\in P_i^*\times P_j^*}\mu_{Q_{ij}}(xy) + \sum_{Q_{ij}\in\mathcal{E}}\sum_{xy\in P_i^*\times P_j^*}\nu_{Q_{ij}}(xy) = \sum_{Q_{ij}\in\mathcal{E}}|P_i^*\times P_j^*|.$$

Proof. (i) Since H is self-complementary, there exists an isomorphism $\varphi\colon H \to H^c$. Let $1 \le i \le |\mathcal{V}|$. Using Lemma 1 and the fact that φ is a bijection (by Definition 12(i) and Lemma 1, φ maps the finite set P_i^* injectively into $(P_i^c)^* = P_i^*$, hence onto P_i^*), we obtain

$$\sum_{x\in P_i^*}\mu_{P_i}(x) = \sum_{x\in P_i^*}\mu^c_{P_i}(\varphi(x)) = \sum_{x\in P_i^*}(1-\nu_{P_i}(\varphi(x))) = |P_i^*| - \sum_{x\in P_i^*}\nu_{P_i}(\varphi(x)) = |P_i^*| - \sum_{x\in P_i^*}\nu_{P_i}(x).$$

Hence, $\sum_{x\in P_i^*}\mu_{P_i}(x)+\sum_{x\in P_i^*}\nu_{P_i}(x) = |P_i^*|$. Summing over i and using that the supports P_i^* are pairwise disjoint with union X (Definition 7), we obtain $\sum_{i=1}^{|\mathcal{V}|}|P_i^*| = |X|$, which is item (i).

(ii) Let $1 \le i \le |\mathcal{D}|$. Then, since φ is a bijection,

$$\sum_{xy\in P_i^*\times P_i^*}\mu_{R_i}(xy) = \sum_{xy\in P_i^*\times P_i^*}\mu^c_{R_i}(\varphi(x)\varphi(y)) = \sum_{xy\in P_i^*\times P_i^*}(1-\nu_{R_i}(\varphi(x)\varphi(y))) = |P_i^*\times P_i^*| - \sum_{xy\in P_i^*\times P_i^*}\nu_{R_i}(xy).$$

Hence, $\sum_{xy\in P_i^*\times P_i^*}\mu_{R_i}(xy)+\sum_{xy\in P_i^*\times P_i^*}\nu_{R_i}(xy) = |P_i^*\times P_i^*|$. Summing over $i = 1,\dots,|\mathcal{D}|$ gives item (ii).

(iii) The proof is similar to that of (ii): for each $Q_{ij}\in\mathcal{E}$ one obtains $\sum_{xy\in P_i^*\times P_j^*}\big(\mu_{Q_{ij}}(xy)+\nu_{Q_{ij}}(xy)\big) = |P_i^*\times P_j^*|$, and summing over $Q_{ij}\in\mathcal{E}$ gives item (iii). □

4 Valued Density of Pythagorean Fuzzy Superhypergraphs

In this section, for every $k\in\mathbb{N}$ and every RPFS H, we introduce the k-density of Pythagorean fuzzy links, the k-density of Pythagorean fuzzy superedges, and the k-density of H. We investigate the k-density of (c,c')-constant complete RPFSs and of self-complementary RPFSs. The Pythagorean fuzzy link-degree and superedge-degree of a vertex of an RPFS are introduced, which leads to the concept of (r,s)-regular RPFSs for $r,s\in\mathbb{R}$. Finally, we compute the k-density of (r,s)-regular RPFSs and investigate their properties.

Definition 14. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be an RPFS on X and let $k \in \mathbb{N}$. The *k-density of the Pythagorean fuzzy links $\mathcal{D}$* is

$$D^*(H,\mathcal{D}) = \left(\frac{k\sum_{i=1}^{d}\sum_{xy\in R_i^*}\mu_{R_i}(xy)}{\sum_{i=1}^{d}\sum_{xy\in R_i^*}(\mu_{P_i}(x)\wedge\mu_{P_i}(y))}, \frac{k\sum_{i=1}^{d}\sum_{xy\in R_i^*}\nu_{R_i}(xy)}{\sum_{i=1}^{d}\sum_{xy\in R_i^*}(\nu_{P_i}(x)\vee\nu_{P_i}(y))} \right), \tag{1}$$

the *k-density of the Pythagorean fuzzy superedges $\mathcal{E}$* is

$$D^*(H,\mathcal{E}) = \left(\frac{k\sum_{Q_{ij}\in\mathcal{E}}\sum_{xy\in Q_{ij}^*}\mu_{Q_{ij}}(xy)}{\sum_{Q_{ij}\in\mathcal{E}}\sum_{xy\in Q_{ij}^*}(\mu_{P_i}(x)\wedge\mu_{P_j}(y))}, \frac{k\sum_{Q_{ij}\in\mathcal{E}}\sum_{xy\in Q_{ij}^*}\nu_{Q_{ij}}(xy)}{\sum_{Q_{ij}\in\mathcal{E}}\sum_{xy\in Q_{ij}^*}(\nu_{P_i}(x)\vee\nu_{P_j}(y))} \right), \tag{2}$$

and the *k-density of H* is the pair $D^*(H) = \big(D^*(H,\mathcal{D}), D^*(H,\mathcal{E})\big)$.

Remark 1. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be an RPFS on X, let $k \in \mathbb{N}$, and let $\alpha, \beta \in \mathbb{R}^{\geq 0}$. The importance of the number k is that, in order to increase or decrease the k-density of the Pythagorean fuzzy links to a desired value (α, β), we solve the inequality $D^*(H,\mathcal{D}) \geq (\alpha, \beta)$ or $D^*(H,\mathcal{D}) \leq (\alpha, \beta)$, respectively. The same applies to the k-density of the Pythagorean fuzzy superedges. Here, $(r, s) \leq (\alpha, \beta)$ means $r \leq \alpha$ and $s \leq \beta$.

Example 5. (i) Consider the complex hypernetwork H in Figure 1. Computations show that

$$D^*(H,\mathcal{D}) = \left(\frac{15k}{22}, \frac{415k}{360}\right), \qquad D^*(H,\mathcal{E}) = \left(\frac{16k}{21}, \frac{462k}{400}\right),$$
$$D^*(H) = \left(\left(\frac{15k}{22}, \frac{415k}{360}\right), \left(\frac{16k}{21}, \frac{462k}{400}\right)\right).$$

Assume that k is the impact factor of the school staff's assistance, that $(\alpha, \beta) = (1, 3)$, and that we want $D^*(H,\mathcal{D}) \leq (\alpha, \beta)$. Then $\left(\frac{15k}{22}, \frac{415k}{360}\right) \leq (1, 3)$, and so $k = 1$. Hence, $\left(\frac{15}{22}, \frac{415}{360}\right)$ is the density of the Pythagorean fuzzy links with respect to (α, β).

(ii) Consider the complete RPFS in Example 2 (Figure 2). Then $D^*(H) = \big((k, k), (k, k)\big)$.

For $x, y \in X$, if $xy \notin R_i^*$ we set $\mu_{R_i}(xy) = \nu_{R_i}(xy) = 0$, and if $xy \notin Q_{ij}^*$ we set $\mu_{Q_{ij}}(xy) = \nu_{Q_{ij}}(xy) = 0$. With this convention, we have the following result.

Corollary 2. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be an RPFS on X. Then

(i)

$$D^*(H,\mathcal{D}) = \left(\frac{k\sum_{x,y\in X}\mu_{R_i}(xy)}{\sum_{x,y\in X}(\mu_{P_i}(x)\wedge\mu_{P_i}(y))}, \frac{k\sum_{x,y\in X}\nu_{R_i}(xy)}{\sum_{x,y\in X}(\nu_{P_i}(x)\vee\nu_{P_i}(y))} \right).$$

(ii)

$$D^*(H,\mathcal{E}) = \left(\frac{k\sum_{x,y\in X}\mu_{Q_{ij}}(xy)}{\sum_{x,y\in X}\big(\mu_{P_i}(x)\wedge\mu_{P_j}(y)\big)}, \frac{k\sum_{x,y\in X}\nu_{Q_{ij}}(xy)}{\sum_{x,y\in X}\big(\nu_{P_i}(x)\vee\nu_{P_j}(y)\big)} \right).$$

In the following, we investigate the k-density of (c,c')-constant complete RPFSs.

Theorem 3. Let $H=(\mathcal{V},\mathcal{E},\mathcal{D})$ be a (c,c')-constant complete RPFS on X. Then

(i) $D^*(H,\mathcal{D})=D^*(H,\mathcal{E})=(k,k)$,

(ii) $D^*(H)=\big((k,k),(k,k)\big)$.

Proof. (i) Since H is a (c,c')-constant complete RPFS on X, for all $xy\in P_i^*\times P_i^*$ we have $\mu_{R_i}(xy)=\min\{\mu_{P_i}(x),\mu_{P_i}(y)\}=c$ and $\nu_{R_i}(xy)=\max\{\nu_{P_i}(x),\nu_{P_i}(y)\}=c'$. Hence, by (1),

$$D^*(H,\mathcal{D}) = \left(\frac{k\sum_{i=1}^d\sum_{xy\in R_i^*}\big(\mu_{P_i}(x)\wedge\mu_{P_i}(y)\big)}{\sum_{i=1}^d\sum_{xy\in R_i^*}\big(\mu_{P_i}(x)\wedge\mu_{P_i}(y)\big)}, \frac{k\sum_{i=1}^d\sum_{xy\in R_i^*}\big(\nu_{P_i}(x)\vee\nu_{P_i}(y)\big)}{\sum_{i=1}^d\sum_{xy\in R_i^*}\big(\nu_{P_i}(x)\vee\nu_{P_i}(y)\big)} \right) = (k,k).$$

In the same way, (2) gives $D^*(H,\mathcal{E})=(k,k)$.

(ii) This follows immediately from (i). □

Example 6. Consider the $(0.5,0.7)$-constant RPFS H on X shown in Figure 3. Then $D^*(H)=\big((k,k),(k,k)\big)$.

Theorem 4. Let $H=(\mathcal{V},\mathcal{E},\mathcal{D})$ be a self-complementary RPFS on X. Then

(i) $D^*(H^c,\mathcal{D}^c)=D^*(H,\mathcal{D})$;

(ii) $D^*(H^c,\mathcal{E}^c)=D^*(H,\mathcal{E})$.

Proof. (i) Since H is self-complementary, there exists an isomorphism $\varphi\colon H\to H^c$. Using Lemma 1, Definition 12, Definition 10, and the fact that φ is a bijection, the first component of $D^*(H,\mathcal{D})$ satisfies

$$\frac{k\sum_{i=1}^d\sum_{xy\in R_i^*}\mu_{R_i}(xy)}{\sum_{i=1}^d\sum_{xy\in R_i^*}\big(\mu_{P_i}(x)\wedge\mu_{P_i}(y)\big)} = \frac{k\sum_{i=1}^d\sum_{xy\in R_i^*}\mu^c_{R_i}(\varphi(x)\varphi(y))}{\sum_{i=1}^d\sum_{xy\in R_i^*}\big(\mu^c_{P_i}(\varphi(x))\wedge\mu^c_{P_i}(\varphi(y))\big)}$$
$$= \frac{k\sum_{i=1}^d\sum_{xy\in R_i^*}\big(1-\nu_{R_i}(\varphi(x)\varphi(y))\big)}{\sum_{i=1}^d\sum_{xy\in R_i^*}\big((1-\nu_{P_i}(\varphi(x)))\wedge(1-\nu_{P_i}(\varphi(y)))\big)}$$

$$= \frac{k\sum_{i=1}^{d}\sum_{xy\in R_i^*}(1-\nu_{R_i}(xy))}{\sum_{i=1}^{d}\sum_{xy\in R_i^*}((1-\nu_{P_i}(x))\wedge(1-\nu_{P_i}(y)))}$$
$$= \frac{k\sum_{i=1}^{d}\sum_{xy\in R_i^*}\mu_{R_i}^c(xy)}{\sum_{i=1}^{d}\sum_{xy\in R_i^*}(\mu_{P_i}^c(x)\wedge\mu_{P_i}^c(y))},$$

which is the first component of $D^*(H^c,\mathcal{D}^c)$. The second components are treated in the same way, using $\nu^c = 1-\mu$ and replacing $\wedge$ by $\vee$. Hence, $D^*(H^c,\mathcal{D}^c) = D^*(H,\mathcal{D})$.

(ii) Similarly, using (2), we obtain $D^*(H^c,\mathcal{E}^c) = D^*(H,\mathcal{E})$. □

Corollary 3. Let $H = (\mathcal{V},\mathcal{E},\mathcal{D})$ be a self-complementary RPFS on X. Then $D^*(H^c) = D^*(H)$.

Definition 15. Let $H = (\mathcal{V},\mathcal{E},\mathcal{D})$ be an RPFS on X and let $x \in P_i^*$. The *Pythagorean fuzzy link-degree* and the *Pythagorean fuzzy superedge-degree* of x are defined, respectively, by

$$\begin{aligned}
\deg_{\mathcal{V}}(x) &= \big(\deg(x,\mathcal{V},\mu_{P_i}), \deg(x,\mathcal{V},\nu_{P_i})\big)\\
&= \Big(\sum_{y\in X,\, y\neq x}\mu_{R_i}(xy), \sum_{y\in X,\, y\neq x}\nu_{R_i}(xy)\Big),\\
\deg_{\mathcal{E}}(x) &= \big(\deg(x,\mathcal{E},\mu_{P_i}), \deg(x,\mathcal{E},\nu_{P_i})\big)\\
&= \Big(\sum_{y\in X,\, y\neq x}\mu_{Q_{ij}}(xy), \sum_{y\in X,\, y\neq x}\nu_{Q_{ij}}(xy)\Big).
\end{aligned}$$

In addition, $\deg(H) = \sum_{x\in X}(\deg_{\mathcal{V}}(x) + \deg_{\mathcal{E}}(x))$ is called the degree of H.

Example 7. For the RPFS H in Figure 1, one can see that $\deg_{\mathcal{V}}(p_4) = (0.3, 0.7)$ and $\deg_{\mathcal{E}}(p_4) = (0.4, 1.5)$.

Definition 16. Let $H = (\mathcal{V},\mathcal{E},\mathcal{D})$ be an RPFS on X and let $r, s \in \mathbb{R}^{>0}$. We say that H is *(r,s)-regular* if $\deg_{\mathcal{V}}(x) = \deg_{\mathcal{E}}(x) = (r,s)$ for all $x \in X$.

Example 8. Consider the RPFS H in Figure 6. Then H is $(0.2, 0.7)$-regular.

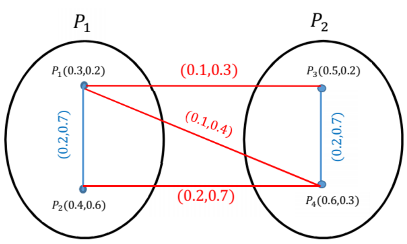

Figure 6: A $(0.2, 0.7)$-regular RPFS.

Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be an (r, s)-regular RPFS on X. For $1 \le i \le d$ and $Q_{ij} \in \mathcal{E}$, put

$$\alpha_i = \sum_{xy \in R_i^*} (\mu_{P_i}(x) \wedge \mu_{P_i}(y)), \qquad \beta_i = \sum_{xy \in R_i^*} (\nu_{P_i}(x) \vee \nu_{P_i}(y)),$$

$$\gamma_{ij} = \sum_{xy \in Q_{ij}^*} (\mu_{P_i}(x) \wedge \mu_{P_j}(y)), \qquad \delta_{ij} = \sum_{xy \in Q_{ij}^*} (\nu_{P_i}(x) \vee \nu_{P_j}(y)).$$

We then have the following result.

Theorem 5. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be an (r, s)-regular RPFS on X, and let $t, t' > 0$ be the constants such that, for all i and j,

$$\sum_{xy \in R_i^*} \mu_{R_i}(xy) = rt, \qquad \sum_{xy \in R_i^*} \nu_{R_i}(xy) = st,$$

$$\sum_{xy \in Q_{ij}^*} \mu_{Q_{ij}}(xy) = rt', \qquad \sum_{xy \in Q_{ij}^*} \nu_{Q_{ij}}(xy) = st'.$$

Then

$$D^*(H) = \left(\left(\frac{k\,d\,r\,t}{\sum_{i=1}^d \alpha_i}, \frac{k\,d\,s\,t}{\sum_{i=1}^d \beta_i} \right), \left(\frac{k\,e\,r\,t'}{\sum_{Q_{ij} \in \mathcal{E}} \gamma_{ij}}, \frac{k\,e\,s\,t'}{\sum_{Q_{ij} \in \mathcal{E}} \delta_{ij}} \right) \right).$$

Proof. Since H is (r, s)-regular, (1) gives

$$D^*(H, \mathcal{D}) = \left(\frac{k \sum_{i=1}^d rt}{\sum_{i=1}^d \alpha_i}, \frac{k \sum_{i=1}^d st}{\sum_{i=1}^d \beta_i} \right) = \left(\frac{k\,d\,r\,t}{\sum_{i=1}^d \alpha_i}, \frac{k\,d\,s\,t}{\sum_{i=1}^d \beta_i} \right).$$

In the same way, (2) gives

$$D^*(H, \mathcal{E}) = \left(\frac{k\,e\,r\,t'}{\sum_{Q_{ij} \in \mathcal{E}} \gamma_{ij}}, \frac{k\,e\,s\,t'}{\sum_{Q_{ij} \in \mathcal{E}} \delta_{ij}} \right).$$

□

Example 9. Consider the RPFS H in Figure 7. Computations show that H is $(0.4, 0.8)$-regular

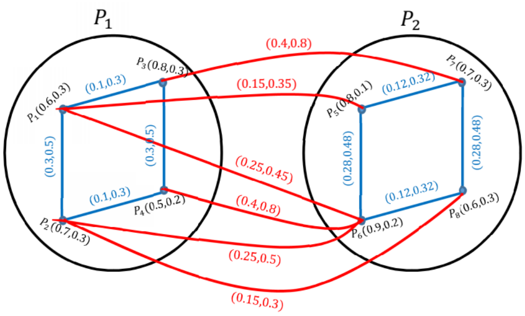

Figure 7: A $(0.4, 0.8)$-regular RPFS.

and

$$D^*(H) = \left(\left(\frac{16k}{49}, \frac{32k}{23} \right), \left(\frac{16k}{37}, \frac{32k}{17} \right) \right),$$

where $t = 2$ and $t' = 4$.

4.1 Balanced Pythagorean Fuzzy Superhypergraphs

In this subsection, we consider an arbitrary RPFS H and introduce its Pythagorean fuzzy subsuperhypergraphs. We present the notion of a balanced RPFS via the valued density of its subsuperhypergraphs and analyze isomorphic balanced RPFSs. Finally, we introduce a class of balanced RPFSs.

Definition 17. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be an RPFS on X, where $\mathcal{V} = \{P_i\}_{i=1}^{v}$, $\mathcal{E} = \{Q_{ij}\}_{i,j\geq 1}^{e}$, and $\mathcal{D} = \{R_i\}_{i=1}^{d}$, and let $G = (\mathcal{U}, \mathcal{F}, \mathcal{C})$ be an RPFS on a non-empty subset $W \subseteq X$, called the *underlying set* of G, where $\mathcal{U} = \{U_i = (\mu_{U_i}, \nu_{U_i})\}_{i=1}^{u}$, $\mathcal{F} = \{F_{ij} = (\mu_{F_{ij}}, \nu_{F_{ij}})\}_{i,j\geq 1}^{f}$, and $\mathcal{C} = \{C_i = (\mu_{C_i}, \nu_{C_i})\}_{i=1}^{c}$. Then G is called a *Pythagorean fuzzy subsuperhypergraph* of H, denoted $G \leq_{\text{sub}} H$, if

(1) $u \leq v$ and $U_i^* \subseteq P_i^*$ for every i;

(2) for every $x \in U_i^*$, $\mu_{U_i}(x) \leq \mu_{P_i}(x)$ and $\nu_{P_i}(x) \leq \nu_{U_i}(x)$;

(3) for every $xy \in C_i^*$, $\mu_{C_i}(xy) \leq \mu_{R_i}(xy)$ and $\nu_{R_i}(xy) \leq \nu_{C_i}(xy)$;

(4) for every $xy \in F_{ij}^*$, $\mu_{F_{ij}}(xy) \leq \mu_{Q_{ij}}(xy)$ and $\nu_{Q_{ij}}(xy) \leq \nu_{F_{ij}}(xy)$.

Example 10. Consider the complex hypernetwork H in Figure 1. Suppose that the students of two classes want to carry out a group project, represented by G in Figure 8, for the academic progress of their school. Then G is a Pythagorean fuzzy subsuperhypergraph of H.

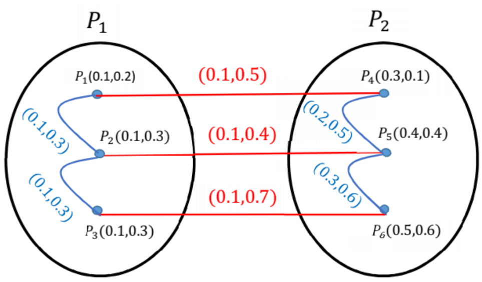

Figure 8: A Pythagorean fuzzy subsuperhypergraph G of H.

Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ and $H' = (\mathcal{V}', \mathcal{E}', \mathcal{D}')$ be RPFSs on X and X', respectively, and let $\varphi\colon X \to X'$ be a bijection. If $G' = (\mathcal{U}', \mathcal{F}', \mathcal{C}') \leq_{\text{sub}} H'$, we define $\varphi^{-1}(G') = (\varphi^{-1}(\mathcal{U}'), \varphi^{-1}(\mathcal{F}'), \varphi^{-1}(\mathcal{C}'))$, where

$$\varphi^{-1}(\mathcal{U}') = \{\varphi^{-1}(U_i') = (\mu_{\varphi^{-1}(U_i')}, \nu_{\varphi^{-1}(U_i')})\}_{i=1}^{u'},$$
$$\varphi^{-1}(\mathcal{F}') = \{\varphi^{-1}(F_{ij}') = (\mu_{\varphi^{-1}(F_{ij}')}, \nu_{\varphi^{-1}(F_{ij}')})\}_{i,j\geq 1}^{f'},$$

$$\varphi^{-1}(\mathcal{C}') = \{\varphi^{-1}(C_i') = (\mu_{\varphi^{-1}(C_i')}, \nu_{\varphi^{-1}(C_i')})\}_{i=1}^{c'},$$

that is, $\mu_{\varphi^{-1}(U_i')}(x) = \mu_{U_i'}(\varphi(x))$ and $\nu_{\varphi^{-1}(U_i')}(x) = \nu_{U_i'}(\varphi(x))$, and analogously for F_{ij}' and C_i', with xy evaluated at $\varphi(x)\varphi(y)$.

Theorem 6. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ and $H' = (\mathcal{V}', \mathcal{E}', \mathcal{D}')$ be RPFSs on X and X', respectively, and let $\varphi \colon X \to X'$ be an isomorphism. If $G' \leq_{\text{sub}} H'$ with underlying set $W' \subseteq X'$, then $\varphi^{-1}(G') \leq_{\text{sub}} H$ with underlying set $\varphi^{-1}(W') \subseteq X$.

Proof. Since $\emptyset \neq W' \subseteq X'$, we have $\emptyset \neq \varphi^{-1}(W') \subseteq X$, and $\varphi^{-1}(\mathcal{U}')$, $\varphi^{-1}(\mathcal{F}')$, and $\varphi^{-1}(\mathcal{C}')$ are families of PFSs on $\varphi^{-1}(W')$. Because $G' \leq_{\text{sub}} H'$, we have $u' \leq v'$ and $U_i'^{*} \subseteq P_i'^{*}$; because φ is an isomorphism, $v' = v$, so $u' \leq v$.

Supports. Let $x \in (\varphi^{-1}(U_i'))^{*}$. Then $\mu_{U_i'}(\varphi(x)) > 0$ and $\nu_{U_i'}(\varphi(x)) < 1$, that is, $\varphi(x) \in U_i'^{*} \subseteq P_i'^{*}$. Since φ is an isomorphism, $\mu_{P_i}(x) = \mu_{P_i'}(\varphi(x)) > 0$ and $\nu_{P_i}(x) = \nu_{P_i'}(\varphi(x)) < 1$, that is, $x \in P_i^{*}$. Hence $(\varphi^{-1}(U_i'))^{*} \subseteq P_i^{*}$.

Supervertices. For $x \in \varphi^{-1}(W')$,

$$\mu_{\varphi^{-1}(U_i')}(x) = \mu_{U_i'}(\varphi(x)) \leq \mu_{P_i'}(\varphi(x)) = \mu_{P_i}(x),$$
$$\nu_{\varphi^{-1}(U_i')}(x) = \nu_{U_i'}(\varphi(x)) \geq \nu_{P_i'}(\varphi(x)) = \nu_{P_i}(x).$$

Links. For $x, y \in \varphi^{-1}(W')$,

$$\mu_{\varphi^{-1}(C_i')}(xy) = \mu_{C_i'}(\varphi(x)\varphi(y)) \leq \mu_{R_i'}(\varphi(x)\varphi(y)) = \mu_{R_i}(xy),$$
$$\nu_{\varphi^{-1}(C_i')}(xy) = \nu_{C_i'}(\varphi(x)\varphi(y)) \geq \nu_{R_i'}(\varphi(x)\varphi(y)) = \nu_{R_i}(xy).$$

Superedges. For $x, y \in \varphi^{-1}(W')$,

$$\mu_{\varphi^{-1}(F_{ij}')}(xy) = \mu_{F_{ij}'}(\varphi(x)\varphi(y)) \leq \mu_{Q_{ij}'}(\varphi(x)\varphi(y)) = \mu_{Q_{ij}}(xy),$$
$$\nu_{\varphi^{-1}(F_{ij}')}(xy) = \nu_{F_{ij}'}(\varphi(x)\varphi(y)) \geq \nu_{Q_{ij}'}(\varphi(x)\varphi(y)) = \nu_{Q_{ij}}(xy).$$

Since $\varphi^{-1}(\mathcal{U}')$, $\varphi^{-1}(\mathcal{F}')$, and $\varphi^{-1}(\mathcal{C}')$ are families of PFSs on $\varphi^{-1}(W')$, we conclude that $\varphi^{-1}(G') \leq_{\text{sub}} H$ with underlying set $\varphi^{-1}(W') \subseteq X$. $\square$

Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ and $H' = (\mathcal{V}', \mathcal{E}', \mathcal{D}')$ be RPFSs on X and X', respectively, and let $\varphi \colon X \to X'$ be a bijection. If $G = (\mathcal{U}, \mathcal{F}, \mathcal{C}) \leq_{\text{sub}} H$, we define $\varphi(G) = (\varphi(\mathcal{U}), \varphi(\mathcal{F}), \varphi(\mathcal{C}))$, where

$$\varphi(\mathcal{U}) = \{\varphi(U_i) = (\mu_{\varphi(U_i)}, \nu_{\varphi(U_i)})\}_{i=1}^{u},$$
$$\varphi(\mathcal{F}) = \{\varphi(F_{ij}) = (\mu_{\varphi(F_{ij})}, \nu_{\varphi(F_{ij})})\}_{i,j\geq 1}^{f},$$
$$\varphi(\mathcal{C}) = \{\varphi(C_i) = (\mu_{\varphi(C_i)}, \nu_{\varphi(C_i)})\}_{i=1}^{c},$$

with $\mu_{\varphi(U_i)}(y) = \bigvee_{y=\varphi(x)} \mu_{U_i}(x)$ and $\nu_{\varphi(U_i)}(y) = \bigvee_{y=\varphi(x)} \nu_{U_i}(x)$, and analogously for F_{ij} and C_i. Since φ is a bijection, each supremum is taken over a single element. Similarly to Theorem 6, we can prove the following theorem.

Theorem 7. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ and $H' = (\mathcal{V}', \mathcal{E}', \mathcal{D}')$ be RPFSs on X and X', respectively, and let $\varphi\colon X \to X'$ be an isomorphism. If $G \leq_{\text{sub}} H$ with underlying set $W \subseteq X$, then $\varphi(G) \leq_{\text{sub}} H'$ with underlying set $\varphi(W) \subseteq X'$.

Proof. Since $\emptyset \neq W \subseteq X$, we have $\emptyset \neq \varphi(W) \subseteq X'$, and $\varphi(\mathcal{U})$, $\varphi(\mathcal{F})$, and $\varphi(\mathcal{C})$ are families of PFSs on $\varphi(W)$. Since $G \leq_{\text{sub}} H$, we have $u \leq v$ and $U_i^* \subseteq P_i^*$.

Supports. Let $y \in (\varphi(U_i))^*$. Since φ is a bijection, $y = \varphi(x)$ for a unique $x \in W$, and $\mu_{U_i}(x) > 0$, $\nu_{U_i}(x) < 1$; that is, $x \in U_i^* \subseteq P_i^*$. Since φ is an isomorphism, $\mu_{P_i'}(y) = \mu_{P_i}(x) > 0$ and $\nu_{P_i'}(y) = \nu_{P_i}(x) < 1$, so $(\varphi(U_i))^* \subseteq P_i'^*$.

Supervertices. Let $y = \varphi(x) \in \varphi(W)$. Then

$$\mu_{\varphi(U_i)}(y) = \mu_{U_i}(x) \leq \mu_{P_i}(x) = \mu_{P_i'}(y),$$
$$\nu_{\varphi(U_i)}(y) = \nu_{U_i}(x) \geq \nu_{P_i}(x) = \nu_{P_i'}(y).$$

Links. Let $x' = \varphi(x)$ and $y' = \varphi(y)$ be in $\varphi(W)$. Then

$$\mu_{\varphi(C_i)}(x'y') = \mu_{C_i}(xy) \leq \mu_{R_i}(xy) = \mu_{R_i'}(x'y'),$$
$$\nu_{\varphi(C_i)}(x'y') = \nu_{C_i}(xy) \geq \nu_{R_i}(xy) = \nu_{R_i'}(x'y').$$

Superedges. With the same x' and y',

$$\mu_{\varphi(F_{ij})}(x'y') = \mu_{F_{ij}}(xy) \leq \mu_{Q_{ij}}(xy) = \mu_{Q_{ij}'}(x'y'),$$
$$\nu_{\varphi(F_{ij})}(x'y') = \nu_{F_{ij}}(xy) \geq \nu_{Q_{ij}}(xy) = \nu_{Q_{ij}'}(x'y').$$

Since $\varphi(\mathcal{U})$, $\varphi(\mathcal{F})$, and $\varphi(\mathcal{C})$ are families of PFSs on $\varphi(W)$, we conclude that $\varphi(G) \leq_{\text{sub}} H'$ with underlying set $\varphi(W) \subseteq X'$. □

Definition 18. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be an RPFS on X. Then H is called *balanced* if $D^*(G) \preceq D^*(H)$ for every $G \leq_{\text{sub}} H$. Here, for $D^*(G) = \big((\alpha_1, \beta_1), (\alpha_1', \beta_1')\big)$ and $D^*(H) = \big((\alpha_2, \beta_2), (\alpha_2', \beta_2')\big)$, the relation $D^*(G) \preceq D^*(H)$ means

$$\alpha_1 \leq \alpha_2, \quad \alpha_1' \leq \alpha_2', \quad \beta_1 \geq \beta_2, \quad \beta_1' \geq \beta_2'.$$

Example 11. Consider the RPFS $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ in Figure 1 and the RPFS G in Figure 8, which is a Pythagorean fuzzy subsuperhypergraph of H. Computations show that

$$D^*(G) = \Big(\Big(\frac{7k}{9}, \frac{17k}{16}\Big), \Big(k, \frac{16k}{12}\Big)\Big),$$

and $D^*(G) \not\preceq D^*(H)$ (see Example 5). This means that, although the number of students and the levels of intelligence, perseverance, understanding, cooperation, interaction, and competition decreased, the density increased. Thus, the complex hypernetwork H is not balanced.

Theorem 8. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be a (c, c')-constant complete RPFS on X. Then H is balanced.

Proof. Let $G = (\mathcal{U}, \mathcal{F}, \mathcal{C}) \leq_{\text{sub}} H$ with underlying set $Y \subseteq X$. By Theorem 3, $D^*(H) = \big((k,k),(k,k)\big)$. Since G is an RPFS, we have $\mu_{C_i}(xy) \leq \mu_{U_i}(x) \wedge \mu_{U_i}(y)$ and $\nu_{C_i}(xy) \geq \nu_{U_i}(x) \vee \nu_{U_i}(y)$ for every $xy \in C_i^*$. Hence, by (1),

$$D^*(G,\mathcal{C}) = \left(\frac{k\sum_{i=1}^{c}\sum_{xy\in C_i^*}\mu_{C_i}(xy)}{\sum_{i=1}^{c}\sum_{xy\in C_i^*}\big(\mu_{U_i}(x)\wedge\mu_{U_i}(y)\big)}, \frac{k\sum_{i=1}^{c}\sum_{xy\in C_i^*}\nu_{C_i}(xy)}{\sum_{i=1}^{c}\sum_{xy\in C_i^*}\big(\nu_{U_i}(x)\vee\nu_{U_i}(y)\big)} \right) \preceq (k,k) = D^*(H,\mathcal{D}).$$

Similarly, since $\mu_{F_{ij}}(xy) \leq \mu_{U_i}(x) \wedge \mu_{U_j}(y)$ and $\nu_{F_{ij}}(xy) \geq \nu_{U_i}(x) \vee \nu_{U_j}(y)$ for every $xy \in F_{ij}^*$, (2) gives

$$D^*(G,\mathcal{F}) \preceq (k,k) = D^*(H,\mathcal{E}).$$

Therefore, $D^*(G) \preceq D^*(H)$, and H is balanced. □

Example 12. Consider the RPFS K_6 in Figure 3 and the RPFS G in Figure 9. By Theorem 8, K_6 is balanced. Computations show that

$$D^*(G) = \left(\left(\frac{10k}{12}, k\right), \left(\frac{17k}{22}, k\right)\right).$$

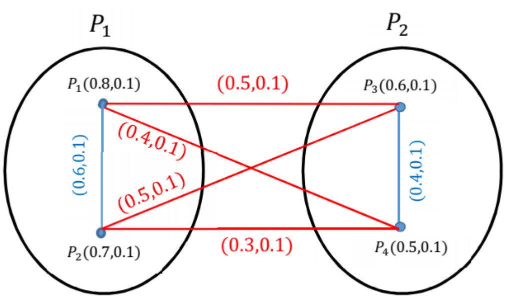

Figure 9: A Pythagorean fuzzy subsuperhypergraph G of K_6.

Proposition 1. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ and $H' = (\mathcal{V}', \mathcal{E}', \mathcal{D}')$ be isomorphic RPFSs on X and X', respectively. Then

(i)
$$\Big(\sum_{xy\in R_i^*}\mu_{R_i}(xy), \sum_{xy\in R_i^*}\nu_{R_i}(xy)\Big) = \Big(\sum_{x'y'\in R_i'^*}\mu_{R_i'}(x'y'), \sum_{x'y'\in R_i'^*}\nu_{R_i'}(x'y')\Big);$$

(ii)

$$\Big(\sum_{xy\in Q_{ij}^*}\mu_{Q_{ij}}(xy),\ \sum_{xy\in Q_{ij}^*}\nu_{Q_{ij}}(xy)\Big)=\Big(\sum_{x'y'\in Q_{ij}'^*}\mu_{Q_{ij}'}(x'y'),\ \sum_{x'y'\in Q_{ij}'^*}\nu_{Q_{ij}'}(x'y')\Big).$$

Proof. (i) Since $H\cong H'$, there exists a bijection $\varphi\colon X\to X'$ satisfying the conditions of Definition 12. By condition (i) of that definition, φ maps P_i^* onto $P_i'^*$; hence, it maps $P_i^*\times P_i^*$ bijectively onto $P_i'^*\times P_i'^*$. Using condition (ii) of Definition 12 and the convention introduced before Corollary 2, we obtain

$$\sum_{x'y'\in P_i'^*\times P_i'^*}\mu_{R_i'}(x'y')=\sum_{xy\in P_i^*\times P_i^*}\mu_{R_i'}(\varphi(x)\varphi(y))=\sum_{xy\in P_i^*\times P_i^*}\mu_{R_i}(xy),$$

and the same holds with ν in place of μ.

(ii) The proof is similar to that of (i), using condition (iii) of Definition 12. □

Theorem 9. Let $H=(\mathcal{V},\mathcal{E},\mathcal{D})$ and $H'=(\mathcal{V}',\mathcal{E}',\mathcal{D}')$ be isomorphic RPFSs on X and X', respectively. If H is balanced, then H' is balanced.

Proof. Let $G'=(\mathcal{U}',\mathcal{F}',\mathcal{C}')\leq_{\text{sub}}H'$ with underlying set $W'\subseteq X'$, and let $\varphi\colon X\to X'$ be an isomorphism. By Theorem 6, $\varphi^{-1}(G')\leq_{\text{sub}}H$ with underlying set $\varphi^{-1}(W')\subseteq X$. By the substitution $x'=\varphi(x)$, $y'=\varphi(y)$ and the definitions of $\varphi^{-1}(C_i')$ and $\varphi^{-1}(U_i')$, we have

$$\sum_{x'y'\in C_i'^*}\mu_{C_i'}(x'y')=\sum_{xy\in(\varphi^{-1}(C_i'))^*}\mu_{\varphi^{-1}(C_i')}(xy),$$

$$\sum_{x'y'\in C_i'^*}\big(\mu_{U_i'}(x')\wedge\mu_{U_i'}(y')\big)=\sum_{xy\in(\varphi^{-1}(C_i'))^*}\big(\mu_{\varphi^{-1}(U_i')}(x)\wedge\mu_{\varphi^{-1}(U_i')}(y)\big),$$

and the same identities hold with ν in place of μ and $\vee$ in place of $\wedge$. Consequently,

$$D^*(G',\mathcal{C}')=D^*\big(\varphi^{-1}(G'),\varphi^{-1}(\mathcal{C}')\big).$$

Since H is balanced, $D^*(\varphi^{-1}(G'))\preceq D^*(H)$; in particular,

$$D^*(\varphi^{-1}(G'),\varphi^{-1}(\mathcal{C}'))\preceq D^*(H,\mathcal{D}).$$

Moreover, since φ is an isomorphism,

$$\big(\mu_{R_i}(xy),\nu_{R_i}(xy)\big)=\big(\mu_{R_i'}(\varphi(x)\varphi(y)),\nu_{R_i'}(\varphi(x)\varphi(y))\big)$$

and

$$\big(\mu_{P_i}(x),\nu_{P_i}(x)\big)=\big(\mu_{P_i'}(\varphi(x)),\nu_{P_i'}(\varphi(x))\big),$$

the same substitution, together with Proposition 1, gives $D^*(H,\mathcal{D})=D^*(H',\mathcal{D}')$. Combining these relations,

$$D^*(G',\mathcal{C}')\preceq D^*(H',\mathcal{D}').$$

In the same way, $D^*(G',\mathcal{F}')\preceq D^*(H',\mathcal{E}')$. Hence, $D^*(G')\preceq D^*(H')$, and so H' is balanced. □

Based on Theorems 7 and 9, we obtain the following corollary.

Corollary 4. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ and $H' = (\mathcal{V}', \mathcal{E}', \mathcal{D}')$ be isomorphic RPFSs on X and X', respectively. Then H is balanced if and only if H' is balanced.

5 Conclusion and Discussion

In this paper, we combined the concepts of superhypergraphs and Pythagorean fuzzy subsets to design a purposeful complex hypernetwork. The main contributions and findings are summarized as follows.

1. We introduced k-density measures for the links and superedges of an RPFS (Definition 14) and applied them to characterize the density of (c, c')-constant complete and (r, s)-regular RPFSs (Theorems 3 and 5).

2. Using the valued density, we provided an explicit analytical characterization of the balanced property for the class of (c, c')-constant complete RPFSs (Theorem 8).

3. We proved that the balanced property is invariant under isomorphism (Theorem 9), and we established that the structural invariants of subsuperhypergraphs are preserved as well (Theorems 6 and 7).

4. We extended (r, s)-regularity to RPFSs by generalizing the link-degree and superedge-degree to the vertices of a hypernetwork (Definitions 15 and 16).

Limitations. The theoretical framework developed in this work is primarily structural and combinatorial. Practical implementation and experimental validation on real-world large-scale datasets remain unexplored, which limits the immediate applicability of the proposed measures in applied domains such as social network analysis or biological systems. Moreover, the study focuses on the definition and theoretical investigation of density, regularity, and balancedness in RPFSs; the computational complexity and algorithmic efficiency of computing these measures for large-scale Pythagorean fuzzy superhypergraphs have not been addressed, and the scalability of the proposed approach requires further investigation.

Future work. Future research directions include developing algorithms for computing the k-density in large-scale Pythagorean fuzzy superhypergraphs, extending the framework to other generalizations of fuzzy sets, such as q-rung orthopair fuzzy superhypergraphs, and modeling and optimizing real-world problems based on Pythagorean fuzzy superhypergraphs. These theoretical advancements provide a robust foundation for handling uncertainty in hypergraph-based models across various scientific and engineering disciplines.

Declarations

Availability of Supporting Data

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have influenced the work reported in this paper.

Author Contributions

Mohammad Hamidi: Conceptualization, Investigation, Methodology, Writing – Original Draft. **Shohreh Shirani Nazhvani**: Methodology, Writing – Review & Editing, Investigation, Writing – Original Draft.

Artificial Intelligence Statement

Artificial intelligence (AI) tools, including large language models, were used solely for language editing and improving readability. AI tools were not used to generate research ideas, perform analyses, interpret results, or write the scientific content. All scientific conclusions and intellectual contributions were made exclusively by the authors.

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