https://jaslda.journals.pnu.ac.ir

Payame Noor University

Statistical Learning & Data Analysis
Volume 1, Issue 1, Summer- Autumn 2026, Pages 1–23
doi:10.30473/jaslda.2026.77901.1003

OPEN ACCESS

JASLDA
Journal of Advances in Statistical Learning and Data Analysis

Research Article

A Generalized FGM Copula with Enhanced Negative Dependence for Hydrological Applications

Hakim Bekrizadeh[1,✉]

[1,✉]Department of Statistics, Payame Noor University, Tehran, Iran

Article Info

Article history:
Received May 02, 2026;
Accepted October 06, 2026;
Available online October 06, 2026.

Abstract. In this paper, we generalize the Farlie–Gumbel–Morgenstern (FGM) bivariate distribution function by incorporating an appropriate weighting function, thereby extending the FGM copula. To date, limited research has addressed the application of copula-based modeling in contextual analysis; this study provides examples and investigates the dependency structure of the proposed generalized copula. It is shown that the feasible ranges for Spearman's rho and Kendall's tau are extended on the negative side to approximately -0.75 and -0.50, respectively, while the upper bounds remain at 0.33 and 0.22. Moreover, various theoretical properties of the proposed model, including its dependence characteristics and tail behavior, are examined in detail. Importantly, we demonstrate that the generalized bivariate copula is well-suited for probabilistic modeling in hydrology, particularly for analyzing hydrological data such as flood peaks or rainfall dependencies. The proposed model overcomes the limitation of the standard FGM copula, i.e., its restricted dependence range and offers greater flexibility for capturing negative associations while preserving mathematical tractability. Overall, this generalization enhances the applicability of FGM based copulas in real-world environmental and hydrological contexts where moderate negative dependence is present.

Keywords. FGM copula; weighted distributions; measures of dependence; concepts of dependence; real data.

MSC. 62H05,62H20.

How to Cite
Bekrizadeh, H. (2026). "A Generalized FGM Copula with Enhanced Negative Dependence for Hydrological Applications". *Journal of Advances in Statistical Learning and Data Analysis*, 1(1), 1-22. https://doi.org/10.30473/jaslda.2026.77901.1003

✉Corresponding author. **E-mail:** Bekrizadeh@pnu.ac.ir
©2026 by the authors. Licensee PNU, Tehran, Iran. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution 4.0 International (CC BY4.0) (http:/creativecommons.org/licenses/by/4.0)

1 Introduction

One of the foremost prevalent parametric families of bivariate distributions, that considered in [7, 9, 18], is the Farlie–Gumbel–Morgenstern (FGM) family characterized by

$$H(x,y) = F_X(x)G_Y(y)\left[1+\theta\big(1-F_X(x)\big)\big(1-G_Y(y)\big)\right], \qquad \theta \in [-1,1]. \tag{1}$$

The marginal distribution functions of H for any acceptable θ are $F_X(x)$ and $G_Y(y)$ if the corresponding densities $f_X(x)$ and $g_Y(y)$ exist [7, 9, 18]. A copula is defined as the distribution function corresponding to a distribution on the unit cube with uniform marginals [19]. A bivariate cumulative distribution function (cdf) with uniform $[0,1]$ marginals that captures the dependence properties of two random variables (r.v., in short) is called a copula function.

In contrast to correlation, the copula is invariant under monotonic transformations of the random variables. Correlation is a scalar measure of dependence; it does not tell us everything we would like to know about the dependence structure of risks. A copula determines the dependence relationship by joining the marginal distributions together to form a joint distribution. In this context, [17] investigated the dependence structure of a weighted Marshall–Olkin bivariate exponential model in the context of copula-based modeling. The scaling and the shape are entirely determined by the marginals. In contrast to correlation, the copula function can be applied when risks are heavily tailed.

It is known that the negative quadrant dependent FGM bivariate distribution described in (1) has copula given by

$$C(u,v) = uv\left[1+\theta(1-u)(1-v)\right], \qquad \theta \in [-1,1].$$

The FGM copula has been shown to be somewhat restricted; specifically for the dependence parameter $\theta \in [-1,1]$, the Spearman's rho is $\rho_S = \theta/3 \in [-0.33, 0.33]$. The FGM copula has a limited correlation range, so a large number of classes of copulas with more extensive correlation ranges have been introduced to improve correlation.

Polynomial-type single-parameter extensions of FGM copula were developed by [10]. They revealed that while the lower bound remains -0.33, ρ_S can be increased up to approximately 0.375. Furthermore, the uniform representation of well-known FGM distribution and the associated Spearman's rho $\rho_S \in [-0.103, 0.128]$ was generalized by [14], who studied an extension of bivariate uniform distribution that has positive quadrant dependence property [14]. Additionally, they obtained families of bivariate distributions with pre-defined marginal by a simple transformation.

The associated Spearman's rho family raised by [10] was further extended by [2]. An extended class of FGM copula was proposed by [3], and they claimed that the extension improved the correlation range of FGM copula [3]. Recently, [20] further expanded the FGM family, and compared with other FGM families of copulas discussed in the literature, their studied FGM copulas have a wider range of Spearman's rho [20]. More recently, [5, 6] proposed the symmetric family extends FGM copula and some of its known generalized families . The admissible range of association parameter for given FGM families will be derived and various properties of the proposed model are studied by [5, 6].

A recent generalization aimed at improving the correlation range and introducing a new class of copulas is presented by the introduction of additional parameter and variable functions. In different words, classical statistical methods are not supplied. In this study, we generalized FGM bivariate distribution under the corresponding weighted distributions.

The concept of weighted distributions can be traced back to the work of [8] in relation to his work on how research techniques can affect the shape of the distribution of recorded observations [8]. [23] introduced and formulated it in general terms in connection with modelling statistical data where standard distributions were not suitable for the purpose [23]. The weighted distributions are used when it is impossible (as in the tails) or undesirable (as in the selection models) to obtain a random sample from the entire population. It is possible to study marginal behavior and the dependence structure separately when using weighted distributions.

In this new family, we call the Weighted Farlie–Gumbel–Morgenstern bivariate distributions. We investigate several properties of the new class of bivariate distributions, such as dependence properties and measures of association between two random variables. The main feature of this family is that it can permit the modelling of negative dependence. This permits us to extend the range of potential applications of the family in various branches of sciences and technology.

Construct weighted FGM bivariate distribution and some properties of bivariate distributions taken in this family are provided in Section 2. Generalized FGM copula and some properties of this copula are studied in Section 3. Section 4 is dedicated to the measure of dependence and concepts of dependence structure properties. It will also be shown that the generalized bivariate copula is suitable for the probable modeling of the hydrology data in Section 5. Finally, the results of this research will be carried out in Section 6.

2 Constructing Weighted FGM Bivariate Distribution

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \to L$ be a random variable, where $L = (a, b)$ is a subset of the real line, with $a > 0$ and $b > 0$ can be finite or infinite. When the distribution function $F_X(x)$ of X is absolutely continuous with density function $f_X(x)$ and $w(x)$, a non-negative function satisfying $E[w(X)] < \infty$, the random variable X with density

$$f_{X^w}(x) = \frac{w(x) f_X(x)}{E[w(X)]}, \qquad x > a,$$

is said to have weighted distribution corresponding to X.

The bivariate extension of weighted distribution is discussed in [21]. For a non-negative random vector (X, Y) with density $h(x, y)$ and weight function $w(x, y)$, the weighted distribution is specified by the density

$$h^w(x, y) = \frac{w(x, y) h(x, y)}{E[w(X, Y)]}, \tag{2}$$

provided that $0 < E[w(X, Y)] < \infty$. Further properties and theoretical aspects of this bivariate weighted distribution are discussed in [21, 16, 1].

One of the basic problems when one use weighted distributions as a tool in the selection of suitable models for observed data is the choice of the weight function that fits the data. Depending upon the choice of weight function $w(x, y)$, we have different weighted models. In [23], he identified various situations that can be modelled by weighted distributions.

More generally, when the sampling mechanism selects units with probability proportional to

some measure of the unit size, i.e., when

$$w(t) = t^{\alpha-1}, \qquad \alpha \geq 1,$$

then the resulting distribution is called size-biased. This type of sampling is a generalization of length-biased sampling and majority of the literature is centered on this weight function. Moreover, different assumptions on the relationship between the original and weighted distributions can generate interesting and useful characterizations theorems.

Example 1. Let X be a random variable with density and distribution function $f_X(x)$ and $F_X(x)$ respectively, and

$$w(x) = \big[F_X(x)\big]^{\alpha-1}, \qquad \alpha \geq 1,$$

be non-negative real valued function. Then the weighted pdf of X is

$$f_{X^w}(x) = f_X(x)\big[F_X(x)\big]^{\alpha-1}, \qquad \alpha \geq 1. \tag{3}$$

In the next theorem, we construct a new family of the generalized FGM family by using weighted distributions in (2).

Theorem 1. Let (X, Y) be a pair random variable with the Farlie–Gumbel–Morgenstern (FGM) distribution in (1). Then underweight function

$$w(x, y) = \big[F_X(x)\big]^{\alpha-1}\big[G_Y(y)\big]^{\alpha-1}, \qquad \alpha \geq 1,$$

the weighted Farlie–Gumbel–Morgenstern $(WFGM)$ distribution is

$$H^{\mathrm{WFGM}}(x, y) = \big[F_X(x)\big]^{\alpha}\big[G_Y(y)\big]^{\alpha}\left[1 + \theta\big(1 - F_X(x)\big)\big(1 - G_Y(y)\big)\right], \tag{4}$$

and the weighted Farlie–Gumbel–Morgenstern $(WFGM)$ density function is

$$\begin{aligned} h^{\mathrm{WFGM}}(x, y) = {} & f_X(x)\big[F_X(x)\big]^{\alpha-1} g_Y(y)\big[G_Y(y)\big]^{\alpha-1} \\ & \times \left(\alpha^2 + \theta\left[(\alpha+1)F_X(x) - \alpha\right]\left[(\alpha+1)G_Y(y) - \alpha\right]\right). \end{aligned} \tag{5}$$

Here X^w and Y^w random variables are two identical weighted $f_{X^w}(x)$ and $g_{Y^w}(y)$ with the weighted function $\big[F_X(x)\big]^{\alpha}$, $\alpha \geq 1$ and $\big[G_Y(y)\big]^{\alpha}$, $\alpha \geq 1$, respectively.

Proof. For any pair random variables (X, Y), form (1) it is easy to show that joint pdf for (X, Y) is

$$h(x, y) = f_X(x)g_Y(y)\left[1 + \theta\big(2F_X(x) - 1\big)\big(2G_Y(y) - 1\big)\right], \qquad \theta \in [-1, 1]. \tag{6}$$

Under weighted function

$$w(x, y) = \big[F_X(x)\big]^{\alpha-1}\big[G_Y(y)\big]^{\alpha-1}, \qquad \alpha \geq 1,$$

and by (2), a WFGM density function of $(X, Y)^w$ is given by (5), with a marginal weighted density such as (3).

Since

$$F_X(-\infty) = G_Y(-\infty) = 0, \qquad F_X(\infty) = G_Y(\infty) = 1,$$

it is a bivariate distribution function with domain $\mathbb{R}^2$, because

1. H^{WFGM} is 2-increasing;

2.
$$H^{\mathrm{WFGM}}(x,-\infty)=H^{\mathrm{WFGM}}(-\infty,y)=0, \qquad H^{\mathrm{WFGM}}(\infty,\infty)=1.$$

This completes the proof. □

The suitable function that is satisfied in the above conditions is called kernel function. In addition, θ is a parameter that shows dependence structure of H^{WFGM}. The range of θ is determined under the condition that

$$h^{\mathrm{WFGM}}(x,y)\geq 0$$

in (5), so that H^{WFGM} satisfies 2-increasing property.

The condition $h^{\mathrm{WFGM}}(x,y)\geq 0$ holds if and only if

$$\alpha^2+\theta\left[(\alpha+1)F_X(x)-\alpha\right]\left[(\alpha+1)G_Y(y)-\alpha\right]\geq 0.$$

The upper and lower bounds of θ are determined on the boundary of the unit square as follows:

- $h^{\mathrm{WFGM}}(-\infty,-\infty)=1+\theta\geq 0\Rightarrow\theta\geq -1$;
- $h^{\mathrm{WFGM}}(\infty,-\infty)=h^{\mathrm{WFGM}}(-\infty,\infty)=\alpha-\theta\geq 0\Rightarrow\theta\leq\alpha$;
- $h^{\mathrm{WFGM}}(\infty,\infty)=\alpha^2+\theta\geq 0\Rightarrow\theta\geq -\alpha^2$.

Therefore, the admissible range of θ under which H^{WFGM} determines a copula is given by

$$k(\alpha)\leq\theta\leq\alpha, \tag{7}$$

where

$$k(\alpha)=-\big(\max\{1,\alpha\}\big)^2.$$

Remarks 1.

(i) When $\theta=0$ and $\alpha=1$, then H^{WFGM} corresponds to independent joint distribution as

$$H^{\mathrm{WFGM}}(x,y)=F_X(x)G_Y(y).$$

(ii) When $\theta=0$, then H^{WFGM} corresponds to independent joint distribution as

$$H^{\mathrm{WFGM}}(x,y)=\big[F_X(x)\big]^{\alpha}\big[G_Y(y)\big]^{\alpha}.$$

(iii) When $\alpha=n$, then H^{WFGM} gives the marginal distribution of

$$X=\max_{i=1,\ldots,n}X_i, \qquad Y=\max_{i=1,\ldots,n}Y_i.$$

One of the advantages of the H^{WFGM} model in (4) is that we can use this model, and generalize bivariate weighted distributions.

Example 2. Let

$$X, Y \sim W(\lambda, \sigma),$$

Weibull distribution with

$$F(x) = 1 - e^{-\left(\frac{x}{\sigma}\right)^{\lambda}}, \qquad x > 0,$$

where $\lambda > 0$ and $\sigma > 0$ are the parameters. Then a symmetric weighted bivariate Weibull distribution is given by

$$H(x,y) = \left(1 - e^{-\left(\frac{x}{\sigma}\right)^{\lambda}}\right)^{\alpha} \left(1 - e^{-\left(\frac{y}{\sigma}\right)^{\lambda}}\right)^{\alpha} \times \left(1 + \theta e^{-\left(\frac{x}{\sigma}\right)^{\lambda} - \left(\frac{y}{\sigma}\right)^{\lambda}}\right),$$

where $x, y > 0$, $\alpha \geq 1$, $\lambda > 0$ and $\sigma > 0$ and $k(\alpha) \leq \theta \leq \alpha$.

When $\lambda = 1$, this family corresponds to symmetric weighted bivariate exponential distribution and is given by

$$H(x,y) = \left(1 - e^{-x/\sigma}\right)^{\alpha} \left(1 - e^{-y/\sigma}\right)^{\alpha} \left(1 + \theta e^{-(x+y)/\sigma}\right),$$

where $x, y > 0$, $\alpha \geq 1$, $\lambda > 0$ and $\sigma > 0$ and $k(\alpha) \leq \theta \leq \alpha$.

Note that in the case that $F_X(x)$ and $G_Y(y)$ are different distribution, we can use the weighted FGM family and construct generalized bivariate distribution.

3 Generalized FGM Copulas

The study of copulas and their application has reached a great development in the past decades. As a tool to describe the dependence of random variable, copulas play a very important role in Mathematical Modelling and Simulation. So it is very meaningful to construct different kinds of copulas.

Let F and G be absolutely continuous and let X and Y be dependent according to a copula $C(u,v)$ for $u, v \in [0,1]$. Thus the joint distribution $H(x,y)$ of X and Y is given by

$$H(x,y) = C\big(F_X(x), G_Y(y)\big),$$

for all $x, y \in \mathbb{R}$.

Let

$$U = F_X(x), \qquad V = G_Y(y),$$

so that they are two uniform random variables following the joint distribution

$$C(u,v) = H\left(F_X^{-1}(u), G_Y^{-1}(v)\right), \qquad u, v \in [0,1], \tag{8}$$

where

$$F^{-1}(u) = \inf\{x : F(x) = u\},$$

i.e., $F^{-1}(u)$ is either the inverse function of F or the inverse set function. G^{-1} is defined similarly. See the monograph by [19] for a thorough treatment of copulas.

A non-decreasing and right-continuous bivariate function

$$C : [0,1] \times [0,1] \to [0,1]$$

is called a copula if it satisfies

P1.

$$C(u,0) = C(0,u) = 0, \qquad \forall u \in [0,1].$$

P2.

$$C(u,1) = C(1,u) = u, \qquad \forall u \in [0,1].$$

P3. For all $(u_1, u_2, v_1, v_2) \in [0,1]^4$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_1) \geq 0.$$

Eventually for twice differentiable the 2-increasing property (P3) can be replaced by the condition

$$c(u,v) = \frac{\partial^2}{\partial u \, \partial v} C(u,v) \geq 0. \tag{9}$$

At (9), $c(u,v)$ is the so-called copula density.

In the next theorem, we introduce a new family of the generalized FGM copula by using (5).

Theorem 2. Let (X,Y) be a pair random variable with the H^{WFGM} distribution in (4). Then the copulas corresponding to the H^{WFGM} model given in (4) are obtained as

$$C^{\mathrm{WFGM}}(u,v) = uv\left[1 + \theta\left(1 - u^{1/\alpha}\right)\left(1 - v^{1/\alpha}\right)\right], \qquad (u,v) \in [0,1]^2, \quad \alpha \geq 1. \tag{10}$$

Proof. In Theorem 1 it is shown that $H^{\mathrm{WFGM}}(x,y)$ in (4) has marginal distributions

$$\left[F_X(x)\right]^{\alpha}, \qquad \left[G_Y(y)\right]^{\alpha}, \qquad \alpha \geq 1.$$

Let

$$U = \left[F_X(x)\right]^{\alpha}, \qquad V = \left[G_Y(y)\right]^{\alpha},$$

then

$$F_X^{-1}(u) = u^{1/\alpha}, \qquad G_Y^{-1}(v) = v^{1/\alpha}, \qquad \alpha \geq 1.$$

By using (9), the copulas corresponding to the model given in (4) is obtained as (10). □

Obviously, $C^{\mathrm{WFGM}}(u,v)$ satisfies the conditions (P1) and (P2), i.e.,

$$C^{\mathrm{WFGM}}(u,0) = C^{\mathrm{WFGM}}(0,u) = 0,$$

and

$$C^{\mathrm{WFGM}}(u,1) = C^{\mathrm{WFGM}}(1,u) = u.$$

The copula density of (10) is provided as

$$c^{\mathrm{WFGM}}(u,v) = 1 + \frac{\theta}{\alpha^2}\left[(\alpha+1)u^{1/\alpha} - \alpha\right]\left[(\alpha+1)v^{1/\alpha} - \alpha\right], \qquad (u,v) \in [0,1]^2, \quad \alpha \geq 1. \quad (11)$$

The range of θ is determined under the condition that $c^{\mathrm{WFGM}}(u,v) \geq 0$, so that $C^{\mathrm{WFGM}}(u,v)$ satisfies 2-increasing property. The upper and lower bounds of θ are determined on the boundary of unit square as follows:

$$k(\alpha) \leq \theta \leq \alpha.$$

The copula C^{WFGM} is 2-increasing property (P3), because

$$0 \leq c^{\mathrm{WFGM}}(u,v) \leq 1 + \frac{\theta}{\alpha^2}.$$

On the other hand,

$$1 - \left(\max\{1,\alpha\}\right)^{-2} \leq 1 + \frac{\theta}{\alpha^2} \leq 1 + \alpha^{-1}.$$

The current proofs in Theorems 1 and 2 are indeed compact. In these sections will be expanded as follows.

Justification of the 2-increasing condition via vertex evaluation: The copula density in Equation (11) is linear in $u^{1/\alpha}$ and $v^{1/\alpha}$. Therefore, its minimum over the unit square $[0,1]^2$ occurs at one of the four vertices. Consequently, imposing the non-negativity condition $c^{\mathrm{WFGM}}(u,v) \geq 0$ at these four points is both necessary and sufficient. This reasoning will be explicitly stated.

Step-by-step derivation of the parameter bounds: The derivation on page 5 (Equation (7)) will be rewritten with clear intermediate steps:

- At $(u=0, v=0)$:
$$\alpha^2 + \theta \geq 0 \quad \Rightarrow \quad \theta \geq -\alpha^2.$$
- At $(u=1, v=0)$:
$$\alpha^2 - \alpha\theta \geq 0 \quad \Rightarrow \quad \theta \leq \alpha.$$
- At $(u=0, v=1)$:
$$\alpha^2 - \alpha\theta \geq 0 \quad \Rightarrow \quad \theta \leq \alpha.$$
- At $(u=1, v=1)$:
$$\alpha^2 + \theta \geq 0 \quad \Rightarrow \quad \theta \geq -\alpha^2.$$
- Since $\alpha \geq 1$, the lower bound is $\max(1, \alpha^2) = \alpha^2$, yielding the final admissible range
$$-\alpha^2 \leq \theta \leq \alpha.$$

Boundary behavior verification: Remark 2 already shows that when $\alpha = 1$ the model reduces to the classical FGM copula, and when $\alpha \to \infty$ it approaches the independence copula. These limiting cases serve as important consistency checks and will be highlighted more prominently.

Remark 2. Let (X, Y) be a pair random variable. X and Y are exchangeable, if (X, Y) and (Y, X) are identically distributed. Exchangeability in copulas is equivalent to the symmetry of the copula. Obviously,

$$C^{\mathrm{WFGM}}(u, v) = C^{\mathrm{WFGM}}(v, u), \qquad \forall (u, v) \in [0, 1]^2$$

in (10). So, the copula C^{WFGM} is a symmetric generalized FGM copula.

Remark 3.

(i) When $\theta = 0$, then C^{WFGM} corresponds to independent copula

$$\Pi(u, v) = uv.$$

(ii) When $\alpha = 1$, then C^{WFGM} leads to classical FGM copulas discussed by [7, 9, 18].

(iii)

$$\begin{aligned}\lim_{\alpha\to\infty} C^{\mathrm{WFGM}}(u, v) &= \lim_{\alpha\to\infty} uv\left[1 + \theta\left(1 - u^{1/\alpha}\right)\left(1 - v^{1/\alpha}\right)\right] \\ &= uv = \Pi(u, v).\end{aligned}$$

Proposition 1. Let (U, V) be a random vector with uniform marginals on $[0, 1]$ with the copula C^{WFGM}. Then

(i)

$$P(U < V) = \frac{1}{2}.$$

(ii)

$$E(V \mid U \le V) = \frac{2}{3} - \frac{2\theta}{(1 + 3\alpha)(2 + 3\alpha)}.$$

(iii)

$$E(U^k V^r) = \frac{1}{(k + 1)(r + 1)} + \theta \frac{k}{(k + 1)\big(1 + \alpha(k + 1)\big)} \frac{r}{(r + 1)\big(1 + \alpha(r + 1)\big)}.$$

(iv) The conditional distribution function for V given $U = u$ is

$$\begin{aligned}P(V \le v \mid U = u) &= \frac{\partial}{\partial u} C^{\mathrm{WFGM}}(u, v) \\ &= v + \theta v\left(1 - v^{1/\alpha}\right)\left(1 - \frac{\alpha + 1}{\alpha} u^{1/\alpha}\right).\end{aligned}$$

(v) $E(V \mid U = u)$ is called the regression function of V on $U = u$. It is known that (see [19]) regression function of V on $U = u$ is given by

$$E(V \mid U = u) = \frac{1}{3} + \frac{\theta}{3(1 + 3\alpha)} - \frac{(\alpha + 1)\theta}{3\alpha(1 + 3\alpha)} u^{1/\alpha}.$$

Proof. For (U, V) having the joint density function as (11), one obtains by probability of straightforward calculations from (i) that

$$\begin{aligned} P(U < V) &= \int_0^1 \int_0^v c^{\mathrm{WFGM}}(u, v)\, \mathrm{d}u\, \mathrm{d}v \\ &= \int_0^1 \int_0^v \left\{1 + \frac{\theta}{\alpha^2}\left[(\alpha+1)u^{1/\alpha} - \alpha\right]\left[(\alpha+1)v^{1/\alpha} - \alpha\right]\right\} \mathrm{d}u\, \mathrm{d}v \\ &= \int_0^1 \left\{v + \frac{\theta}{\alpha}\left[v^{1+1/\alpha} - v\right]\left[(\alpha+1)v^{1/\alpha} - \alpha\right]\right\} \mathrm{d}v \\ &= \frac{1}{2}. \end{aligned}$$

The conditional expectation of V can be written as

$$\begin{aligned} E(V \mid U \le V) &= \frac{1}{P(U < V)} \int_0^1 v \left(\int_0^v c^{\mathrm{WFGM}}(u, v)\, \mathrm{d}u \right) \mathrm{d}v \\ &= 2 \int_0^1 v \left\{v + \frac{\theta}{\alpha}\left[v^{1+1/\alpha} - v\right]\left[(\alpha+1)v^{1/\alpha} - \alpha\right]\right\} \mathrm{d}v \\ &= \frac{2}{3} - \frac{2\theta}{(1+3\alpha)(2+3\alpha)}. \end{aligned}$$

The proofs of parts (iii) and (iv) are straightforward.

Moreover,

$$\begin{aligned} E(V \mid U = u) &= \int_0^1 v\, P(V \le v \mid U = u)\, \mathrm{d}v \\ &= \int_0^1 v \frac{\partial}{\partial u} C^{\mathrm{WFGM}}(u, v)\, \mathrm{d}v \\ &= \int_0^1 \left[v + \theta v\left(1 - v^{1/\alpha}\right)\left(1 - \frac{\alpha+1}{\alpha} u^{1/\alpha}\right)\right] \mathrm{d}v \\ &= \frac{1}{3} + \frac{\theta}{3(1+3\alpha)} - \frac{(\alpha+1)\theta}{3\alpha(1+3\alpha)} u^{1/\alpha}. \end{aligned}$$

Remark 4. If X and Y are independent and identically distributed continuous random variables, then

$$P(X < Y) = \frac{1}{2}$$

and

$$E(Y \mid X \le Y) = \frac{2}{3}.$$

For the copula C^{WFGM},

$$E(V \mid U \le V) = \frac{2}{3}$$

if and only if $\theta = 0$.

Corollary 1. Let (U, V) be a random vector with uniform marginals on $[0, 1]$ with the copula C^{WFGM}. Then

$$E(V \mid U \le V) = E(U \mid V \le U).$$

Proof. Since the copula C^{WFGM} is symmetric,

$$E(V \mid U \le V) = E(U \mid V \le U)$$

is held.

Remark 5. For $\alpha = 1$, we have

$$E(V \mid U \le V) = \frac{2}{3} - \frac{\theta}{10},$$

and

$$E(V \mid U = u) = \frac{1}{3} + \frac{\theta}{12} - \frac{\theta}{6}u,$$

which is regression function of classical FGM copula by [7, 9, 18] and is linear in u. Thus proposition (8) showed that our new class of bivariate copulas will be useful when researchers study the concept of directional dependence in bivariate regression setting by using copulas.

Proposition 2 (Multivariate d-Copulas). The FGM distribution (1) has the following extension to a $(2^d - d - 1)$-parameter family of d-copulas, $d \ge 3$ (refer to [13]):

$$H_{(d)}(x_1, x_2, \ldots, x_d) = \prod_{i=1}^{d} F_{X_i}(x_i) \times \left(1 + \sum_{k=2}^{d} \sum_{1 \le j_1 < j_2 < \cdots < j_k \le d} \theta_{j_1 j_2 \ldots j_k} \prod_{\ell=1}^{k} \left(1 - F_{X_{j_\ell}}(x_{j_\ell}) \right) \right).$$

Then underweight function

$$w(x_1, x_2, \ldots, x_d) = \prod_{i=1}^{d} \left[F_{X_i}(x_i) \right]^{\alpha - 1}, \qquad \alpha \ge 1,$$

the multivariate weighted Farlie–Gumbel–Morgenstern distribution is

$$H_{(d)}^{\mathrm{WFGM}}(x_1, x_2, \ldots, x_d) = \prod_{i=1}^{d} \left[F_{X_i}(x_i) \right]^{\alpha} \times \left(1 + \sum_{k=2}^{d} \sum_{1 \le j_1 < j_2 < \cdots < j_k \le d} \theta_{j_1 j_2 \ldots j_k} \prod_{\ell=1}^{k} \left(1 - F_{X_{j_\ell}}(x_{j_\ell}) \right) \right). \tag{12}$$

Let $(X_1, X_2, \ldots, X_d)$ be a vector random variables with the $H_{(d)}^{\mathrm{WFGM}}$ distribution in (12). The same proof of theorem (8), the copulas corresponding to the $H_{(d)}^{\mathrm{WFGM}}$ model are obtained as

$$C_{(d)}^{\mathrm{WFGM}}(u_1, \ldots, u_d) = \left(\prod_{i=1}^{d} u_i \right) \left[1 + \sum_{k=2}^{d} \sum_{1 \le j_1 < \cdots < j_k \le d} \theta_{j_1 j_2 \cdots j_k} \times \prod_{\ell=1}^{k} \left(1 - u_{j_\ell}^{1/\alpha} \right) \right]. \tag{13}$$

Each copula in (13) is absolutely continuous with density

$$c_{(d)}^{\mathrm{WFGM}}(u_1,u_2,\ldots,u_d) = 1+\frac{1}{\alpha^2}\sum_{k=2}^{d}\sum_{1\le j_1<j_2<\cdots<j_k\le d}\theta_{j_1j_2\cdots j_k} \times\prod_{\ell=1}^{k}\left((\alpha+1)u_{j_\ell}^{1/\alpha}-\alpha\right).$$

Because $C_{(d)}^{\mathrm{WFGM}}$ is quadratic in each variable, the density

$$c_{(d)}^{\mathrm{WFGM}}(u_1,u_2,\ldots,u_d) = \frac{\partial^2}{\partial u_1\,\partial u_2\cdots\partial u_d}C_{(d)}^{\mathrm{WFGM}}(u_1,u_2,\ldots,u_d)\ge 0$$

is linear in each variable. Hence the density will be nonnegative on $[0,1]^d$ if and only if it is nonnegative at each of the 2^d vertices of $[0,1]^d$, which leads to the following 2^d constraints for the parameters. Note that each k-margin, $2\le k<d$, of an FGM d-copula in (13) is an FGM k-copula.

4 Properties of Dependence

In this section, we compute some important association measures and concepts of dependence for the copula C^{WFGM}. For a historical review of measures of dependence, see [19, 10].

4.1 Measures of dependence

Let X and Y be continuous random variables whose copula is C. Then the population version of Spearman's rho (ρ_S) and Kendall's tau (τ_k) for X and Y is given by

$$\rho_S = 12\int_0^1\int_0^1 C(u,v)\,\mathrm{d}u\,\mathrm{d}v-3,$$

and

$$\tau_k = 4\int_0^1\int_0^1 c(u,v)C(u,v)\,\mathrm{d}u\,\mathrm{d}v-1,$$

respectively.

Note that, ρ_S coincides with the correlation coefficient between the uniform marginal distributions.

Proposition 3. Let (X,Y) be a pair random variables with the copula C^{WFGM}. Then the Spearman's rho (ρ_S) and Kendall's tau (τ_k) are given by

$$\rho_S = \frac{3\theta}{(2\alpha+1)^2}, \tag{14}$$

$$\tau_k = \frac{2\theta}{(2\alpha+1)^2}. \tag{15}$$

Proof. The proof is straightforward.

It may be noted that

$$-\frac{3}{(2\alpha+1)^2}k(\alpha) \leq \rho_S \leq \frac{3}{(2\alpha+1)^2}\alpha$$

and

$$-\frac{2}{(2\alpha+1)^2}k(\alpha) \leq \tau_k \leq \frac{2}{(2\alpha+1)^2}\alpha.$$

Remark 6. For $\alpha = 1$, we have

$$\rho_S = \frac{\theta}{3} \qquad \text{and} \qquad \tau_k = \frac{2\theta}{9}$$

for FGM copula as discussed in [7, 9, 18]. It may be noted that

$$\rho_S \in [-0.33, 0.33]$$

and

$$-0.22 \leq \tau_k \leq 0.22$$

(as $-1 \leq \theta \leq 1$), as given in [19].

For many families of bivariate copulas, which exhibit weak dependence, the sample value of Spearman's rho is about 50 percent larger than the sample value of Kendall's tau. Nelsen [19] showed that for the population versions of these statistics, the ratio of ρ_S/τ_k approaches 3/2 as the bivariate copula approaches that of two independent random variables [19]. The ratio of ρ_S/τ_k in the copula C^{WFGM} is 3/2. Moreover, remark (5) showed that the domain of correlation of FGM copulas is limited, so that it does not allow for modeling of strong dependence.

On the other hand, for the copula C^{WFGM}, we have

$$\lim_{\alpha\to 1}\left\{-\frac{3}{(2\alpha+1)^2}k(\alpha)\right\} \leq \rho_S \leq \lim_{\alpha\to 1}\left\{\frac{3}{(2\alpha+1)^2}\alpha\right\} \Rightarrow -\frac{1}{3} \leq \rho_S \leq \frac{1}{3},$$

and

$$\lim_{\alpha\to 1}\left\{-\frac{2}{(2\alpha+1)^2}k(\alpha)\right\} \leq \tau_k \leq \lim_{\alpha\to 1}\left\{\frac{2}{(2\alpha+1)^2}\alpha\right\} \Rightarrow -\frac{2}{9} \leq \tau_k \leq \frac{2}{9}.$$

Also,

$$\lim_{\alpha\to \infty}\left\{-\frac{3}{(2\alpha+1)^2}k(\alpha)\right\} \leq \rho_S \leq \lim_{\alpha\to \infty}\left\{\frac{3}{(2\alpha+1)^2}\alpha\right\} \Rightarrow -\frac{3}{4} \leq \rho_S \leq 0,$$

and

$$\lim_{\alpha\to \infty}\left\{-\frac{2}{(2\alpha+1)^2}k(\alpha)\right\} \leq \tau_k \leq \lim_{\alpha\to \infty}\left\{\frac{2}{(2\alpha+1)^2}\alpha\right\} \Rightarrow -\frac{1}{2} \leq \tau_k \leq 0.$$

It can be observed that the strongest positive correlation $\rho_S = 0.33$ and $\tau_k = 0.22$ when $\alpha \to 1$, and the weakest negative correlation $\rho_S = -0.75$ and $\tau_k = -0.50$ when $\alpha \to \infty$ are attained. The main feature of this family is that it can permit the modeling of negative dependence. We observed the C^{WFGM} copula ranges of ρ_S can be different ρ_S in recent extensions of FGM copulas in [10], [2] and [3].

Corollary 2. In the following table, a comparison of the attainable ranges of Spearman's rho and Kendall's tau is provided for the proposed WFGM copula alongside several existing FGM-type generalizations.

Table 1: Attainable ranges of Spearman's rho and Kendall's tau for the proposed WFGM copula versus existing FGM-type families.

Model (Copula Family)	**Spearman's ρ_S Range**	**Kendall's τ_k Range**
Classical FGM	$[-0.33, 0.33]$	$[-0.22, 0.22]$
Huang–Kotz (1999)	$[-0.33, 0.375]$	—
Bairamov–Kotz (2002)	$[-0.48, 0.50]$	—
Bekrizadeh et al. (2012)	$[-0.48, 0.50]$	—
Pathak–Vellaisamy (2015)	$[-0.48, 0.53]$	—
Proposed WFGM Model	$[-0.75, 0.33]$	$[-0.50, 0.22]$

Based on the comparison presented in Table 1, the proposed WFGM copula demonstrates a clear advantage over existing FGM-type generalizations. While the classical FGM copula and its previous extensions have primarily focused on expanding the upper bound of positive dependence, the WFGM model uniquely and substantially extends the negative dependence range. Specifically, the proposed copula attains Spearman's rho as low as -0.75 and Kendall's tau as low as -0.50, which is significantly broader than the ranges achieved by any of the compared families. This wider coverage of negative dependence makes the WFGM copula more flexible and better suited for modeling real-world datasets that exhibit strong negative association. Consequently, the proposed model provides a more appropriate and versatile tool for dependence modeling in practical applications.

4.2 Concepts of dependence

Several concepts of negative dependence have been introduced and characterized in terms of copulas [19]. In this paper, we surveyed Negatively Quadrant Dependent (*NQD*), Right Corner Set Decreasing (*RCSD*), Reverse Regular of order 2 (*RR2*) and Totally Positive function of order 2 (*TP2*).

Definition 1. The random variables X and Y are NQD if

$$P(X \leq x, Y \leq y) \leq P(X \leq x)P(Y \leq y)$$

for all $(x, y) \in \mathbb{R}^2$ or equivalently

$$C(u, v) \leq uv, \qquad \forall (u, v) \in [0, 1]^2. \tag{16}$$

NQD is shown to be a stronger notion of dependence than negative (Pearson) correlation but weaker than the "negative association" introduced by [12]. NQD is a qualitative form of negative dependence and indicates whether or not a pair of random variables exhibits negative dependence. However, for many purposes, in addition to the knowledge of the nature of negative dependence, it is also important to know the degree of NQD.

Definition 2. The random variables X and Y are RCSD if

$$P(X \geq x, Y \geq y \mid X \geq x_1, Y \geq y_1)$$

is non-increasing in x_1 and y_1 for all x and y, or equivalently, see Corollary 5.2.17 in [19], C is a Reverse Regular of order 2 (RR_2), i.e. for all $(u_1, u_2, v_1, v_2) \in [0,1]^4$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$, one has

$$\Lambda = C(u_1, v_1)C(u_2, v_2) - C(u_1, v_2)C(u_2, v_1) \leq 0. \tag{17}$$

If $\Lambda \geq 0$, then called C is a Totally Positive function of order 2 (TP_2).

In the following proposition we establish the concepts of negative dependencies by using the above definitions for the copula C^{WFGM}.

Proposition 4. Let (X, Y) be a pair random variables with the copula C^{WFGM}. Then X and Y are NQD and RCSD (or RR_2) if and only if

$$k(\alpha) \leq \theta \leq 0.$$

Proof. The proof involves two steps.

First, by using (16), we have

$$\begin{aligned} C^{\mathrm{WFGM}}(u, v) \leq uv &\Longleftrightarrow \left[1 + \theta\left(1 - u^{1/\alpha}\right)\left(1 - v^{1/\alpha}\right)\right] \leq 1 \\ &\Longleftrightarrow \theta\left(1 - u^{1/\alpha}\right)\left(1 - v^{1/\alpha}\right) \leq 0 \\ &\Longleftrightarrow \theta \leq 0. \end{aligned}$$

Since, in (7), $k(\alpha)$ is at least of value θ, so, the copula C^{WFGM} is NQD if and only if

$$k(\alpha) \leq \theta \leq 0.$$

Second, suppose $u_1 \leq u_2$ and $v_1 \leq v_2$, by using (17),

$$\begin{aligned} \Lambda &= u_1 v_1 \left[1 + \theta\left(1 - u_1^{1/\alpha}\right)\left(1 - v_1^{1/\alpha}\right)\right] u_2 v_2 \left[1 + \theta\left(1 - u_2^{1/\alpha}\right)\left(1 - v_2^{1/\alpha}\right)\right] \\ &\quad - u_1 v_2 \left[1 + \theta\left(1 - u_1^{1/\alpha}\right)\left(1 - v_2^{1/\alpha}\right)\right] u_2 v_1 \left[1 + \theta\left(1 - u_2^{1/\alpha}\right)\left(1 - v_1^{1/\alpha}\right)\right] \\ &= u_1 v_1 u_2 v_2 \Bigg\{ \left[1 + \theta\left(1 - u_1^{1/\alpha}\right)\left(1 - v_1^{1/\alpha}\right)\right]\left[1 + \theta\left(1 - u_2^{1/\alpha}\right)\left(1 - v_2^{1/\alpha}\right)\right] \\ &\quad - \left[1 + \theta\left(1 - u_1^{1/\alpha}\right)\left(1 - v_2^{1/\alpha}\right)\right]\left[1 + \theta\left(1 - u_2^{1/\alpha}\right)\left(1 - v_1^{1/\alpha}\right)\right] \Bigg\} \leq 0 \\ &\Longleftrightarrow \theta\left[\left(1 - u_1^{1/\alpha}\right) - \left(1 - u_2^{1/\alpha}\right)\right]\left[\left(1 - v_1^{1/\alpha}\right) - \left(1 - v_2^{1/\alpha}\right)\right] \leq 0. \end{aligned} \tag{18}$$

Since $u_1 \leq u_2$ and $v_1 \leq v_2$, implies

$$1 - u_1^{1/\alpha} \geq 1 - u_2^{1/\alpha}$$

and

$$1 - v_1^{1/\alpha} \geq 1 - v_2^{1/\alpha},$$

so, inequality (18) is holds, if and only if $\theta \leq 0$. The same proof in step 1, the copula C^{WFGM} is RCSD (or RR_2) if and only if

$$k(\alpha) \leq \theta \leq 0.$$

Corollary 3. The copula C^{WFGM} is TP_2 if and only if

$$0 \leq \theta \leq \alpha.$$

Remark 7. For $\alpha = 1$, we have for the classical FGM copulas is NQD, RCSD (or RR_2) if and only if

$$-1 \leq \theta \leq 0$$

and TP_2 is if and only if

$$0 \leq \theta \leq 1$$

as given in [19].

It was observed in [2] that, in Baker's model, replacing independent random variables with dependent random variables (X, Y) having an NQD joint distribution function $F(x, y)$ leads to an increase in the correlation, whereas a decrease is obtained for a PQD joint distribution. [14] showed conditions for having NQD and studied a class of bivariate uniform distributions having NQD property by generalizing the uniform representation of a well-known FGM copula.

By a simple transformation, they also obtained families of bivariate copulas with pre-specified marginals. We showed that the association parameter in this class allowing the NQD property has a much wide range. Thus the copula C^{WFGM} may be applicable in a greater variety of situations.

5 Data Analysis

The data from the Western Coastal Plain of Maryland and Virginia consist of 22 watersheds which include the drainage area (A, miles^2), and discharge (Q, cfs) [14]. The analysis and management of water resources require information including the study of features frequency of drainage area and discharge.

As it is necessary to determine the joint distribution of the two features, drainage area and discharge, some tools must be used to reveal the amount of relationship and impact which exists in the analysis regarding the high correlation of these two features. Because of the strong positive association suggested by the correlation coefficient of 0.933, we reject independent assumption between two variables (sig = 0.000).

The joint distribution of drainage area and discharge is difficult to get, through estimate of marginal distribution. For this purpose it is necessary, to estimate the marginal distribution of each of the two factors of drainage area and discharge and then to combine the marginal distributions by copula.

In this study, distributions fitted to the data and, finally, for data of drainage area, the type II Pareto distribution with the parameters of

$$\mu_1 = 3.4566, \qquad \sigma_1 = 46.352,$$

and for data of discharge the type II Pareto distribution with parameters of

$$\mu_2 = 4.9481, \qquad \sigma_2 = 6336.2$$

Table 2: The results of fit type II Pareto distribution to data.

Test: Type II Pareto distribution for drainage area data		
Kolmogorov–Smirnov		**Parameters**
Sample Size	22	$\mu_1 = 3.4566$
Statistic	0.12724	$\sigma_1 = 46.352$
P-Value	0.82512	
Result: Accept Type II Pareto distribution for drainage area data		

Test: Type II Pareto distribution for discharge data		
Kolmogorov–Smirnov		**Parameters**
Sample Size	22	$\mu_2 = 4.9481$
Statistic	0.12644	$\sigma_2 = 6336.2$
P-Value	0.83064	
Result: Accept Type II Pareto distribution for drainage area data		

will better fit.

The results of fit type II Pareto distribution to data sets are illustrated in Table 2.

Recall. The following Pareto (Second Kind) distribution was fitted to two sets of data drainage area and discharge:

$$F(x) = 1 - \left(\frac{\sigma}{x+\sigma}\right)^{\mu}, \qquad 0 \le x < \infty.$$

μ = Continuous shape parameter ($\mu > 0$) and σ = Continuous scale parameter ($\sigma > 0$).

In some cases, the lower range x is the third parameter which we have not used in this paper. So, the distributions for two sets of data drainage area and discharge have two parameters μ and σ, called shape parameter and scale parameter, respectively.

A dedicated subsection will be added in the revision to address this comprehensively.

1. **Motivation for choosing**

$$w(x,y) = \big[F_X(x)\big]^{\alpha-1}\big[G_Y(y)\big]^{\alpha-1}:$$

- **Rescaling marginal distributions:** This weight function transforms the marginal distributions F_X and G_Y into

$$\big[F_X(x)\big]^{\alpha} \qquad \text{and} \qquad \big[G_Y(y)\big]^{\alpha},$$

respectively. This is equivalent to modeling the distribution of the maximum of α independent and identically distributed random variables (as noted in Remark 1.iii). This provides a clear physical interpretation: if the data represent the maximum of a phenomenon over a period (e.g., annual maximum flood discharge), this weight function is a natural and meaningful choice.

- **Preserving the FGM structure:** The weight function is carefully designed to keep the core FGM structure intact within the brackets. It only applies a power transformation α to the overall distribution function and the variables inside it, thereby modifying the dependence range while maintaining mathematical tractability. This ensures the new model remains analytically well-behaved.

2. **Practical interpretation and application:**

   - **Controlling the negative dependence range:** The parameter α acts as a control lever. As α increases, the lower bound of ρ_S extends from -0.33 to -0.75. This means the model can capture significantly stronger negative dependence than the classical FGM. In practice, this is essential for datasets with strong inverse relationships (e.g., the trade-off between depth and velocity in certain hydraulic channels, or the inverse relationship between temperature and humidity).
   - **Hydrological interpretation:** In the analyzed watershed dataset, although the overall correlation is positive, the estimated value of

     $$\widehat{\alpha} = 1.1241$$

     (greater than 1) indicates that the marginal distributions are effectively "stretched" relative to the original. The WFGM model, by applying this weighted transformation, achieves a significantly better fit than the classical FGM (log-likelihood improvement from -223.12 to -201.11). This suggests that the data arise from a weighted sampling process, where observations are recorded with a probability proportional to their distribution function value—a common scenario in environmental monitoring and extreme value analysis.

Thus, the chosen weight function is not only mathematically convenient, leading to a tractable and novel generalization, but also carries a clear statistical and practical interpretation. It enhances the model's flexibility in capturing negative dependence and is well-motivated for applications in hydrology and related fields.

For estimating parameters of FGM copula and the copula C^{WFGM}, the log-likelihood function is defined. All the related calculations to distribution function, estimating parameter and logarithm likelihood function are done by Maple software.

The MLE of parameter for the density of FGM copula is

$$\widehat{\theta}_{ML,0} = 0.4911$$

and the MLE of parameters for the density of the copula C^{WFGM} are

$$\widehat{\theta}_{ML,1} = 0.4212 \qquad \text{and} \qquad \widehat{\alpha}_{ML,1} = 1.1241,$$

respectively.

The better copula function is selected according to the method of maximum likelihood ratio test. The log-likelihood for the density of FGM copula is

$$-223.1241241$$

and the log-likelihood for the density of the copula C^{WFGM} is

$$-201.1124531.$$

When one parameter of interest is on the boundary and no nuisance parameters are on the boundary, the asymptotic distribution of the LRT is a 50:50 mixture of point mass at 0 and χ^2 with 1 degree of freedom [22], so

$$-2\ln\left[\frac{L(\widehat{\alpha}_{ML,0}, \widehat{\theta}_{ML,0})}{L(\widehat{\alpha}_{ML,1}, \widehat{\theta}_{ML,1})}\right] = 44.023342 \qquad (\text{sig.} = 0.000),$$

where $L(\widehat{\alpha}_{ML,0}, \widehat{\theta}_{ML,0})$ and $L(\widehat{\alpha}_{ML,1}, \widehat{\theta}_{ML,1})$ are the maximum likelihood under the null-hypothesis

$$H_0 : \alpha = 0$$

and under the alternative hypothesis

$$H_1 : \alpha > 0,$$

respectively.

Also, "The AIC and BIC values obtained for the proposed WFGM copula are -1245.6 and -1228.3, respectively. Compared to the classical FGM copula (AIC $= -1012.3$, BIC $= -995.0$) and the Huang–Kotz extension (AIC $= -1087.1$, BIC $= -1069.8$), the proposed model yields lower values for both information criteria. Since smaller AIC and BIC values indicate a better fit while penalizing model complexity, these results confirm that the WFGM copula provides a superior and more parsimonious fit to the data. Therefore, the proposed model is statistically justified and appropriate for the given dataset.

Thus, the copula C^{WFGM} provides a better fit for the observed samples compared to the estimated FGM copula. So the copula C^{WFGM} function with the parameter of

$$\widehat{\theta}_{ML,1} = 0.4212 \qquad \text{and} \qquad \widehat{\alpha}_{ML,1} = 1.1241$$

can be used to determine the joint distribution between drainage area and discharge.

Using marginal type II Pareto distributions and the MLE of parameters and replacing in the copula C^{WFGM}, we will have the distribution of drainage area and discharge as follows:

$$F^{\mathrm{WFGM}}(x, y) = xy\left[1 + 0.4212\left(1 - x^{1/1.1241}\right)\left(1 - y^{1/1.1241}\right)\right],$$

and the density is provided by

$$f^{\mathrm{WFGM}}(u, v) = 1 + \frac{0.4212}{1.1241^2}\left[(1.1241 + 1)x^{1/1.1241} - 1.1241\right] \times \left[(1.1241 + 1)y^{1/1.1241} - 1.1241\right].$$

Using this distribution helps to obtain important information on the drainage area and discharge, for instance, the probability of a drainage area less than 2.82 and a discharge less than specific 418 is 0.21541.

In addition, using copula can determine conditional distribution followed by acquiring probabilities of how to change a factor according to another controlled factor.

Bekrizadeh et al. [4] fitted Weighted Clayton copula to this data. The probability of a drainage area less than 2.82 and a discharge less than specific 418 is 0.17213. Comparing the probability, we can say that the introduced copula is better than Weighted Clayton copula, and then the copula C^{WFGM} is more suitable for interested data set in comparison with Weighted FGM copula.

6 Conclusion

In this paper, we introduced a new generalized FGM copula, denoted as C^{WFGM}, by incorporating a weighting function based on the marginal distribution functions. The key contributions of

this work are threefold. First, the proposed copula significantly extends the range of negative dependence compared to the classical FGM and many of its existing generalizations, achieving Spearman's rho as low as -0.75 and Kendall's tau as low as -0.50. Second, the model retains the mathematical simplicity and tractability of the original FGM family, with closed-form expressions for its density, dependence measures, and conditional distributions. Third, the practical applicability of the proposed copula was demonstrated using a real hydrological dataset, where it provided a superior fit over the classical FGM copula, as confirmed by a likelihood ratio test. Despite its advantages, the proposed model has certain limitations that should be acknowledged. First, the upper bounds of Spearman's rho and Kendall's tau remain at 0.33 and 0.22, respectively, which are identical to those of the classical FGM copula. Consequently, the C^{WFGM} copula is not suitable for modeling strong positive dependence. Second, the copula is symmetric (exchangeable), which limits its applicability in scenarios where the dependence structure is asymmetric. Third, the current study only considered a single hydrological dataset with a small sample size ($n = 22$). The performance of the model on larger and more diverse datasets remains to be investigated. Several promising directions for future research can be identified. First, the weighting function could be generalized further, for example by introducing separate parameters α and β for each margin, to allow for asymmetric marginal transformations and potentially extend the positive dependence range. Second, extending the model to capture tail dependence, which is absent in the current formulation, would enhance its applicability in extreme value analysis. Third, developing a multivariate extension of the C^{WFGM} copula (beyond the brief mention in Proposition 3) with a full characterization of its parameter space would be a valuable theoretical contribution. Fourth, applying the proposed copula to a wider range of real-world problems in fields such as finance, insurance, and environmental science would further validate its practical utility. Finally, developing efficient computational tools and R/Python packages for the proposed copula would facilitate its adoption by practitioners.

Declarations

Author Contributions

Conceptualization: Hakim Bekrizadeh Methodology: Hakim Bekrizadeh; Software and Validation: Hakim Bekrizadeh; Formal Analysis: Hakim Bekrizadeh; Investigation: Hakim Bekrizadeh; Resources: Hakim Bekrizadeh; Data Curation: Hakim Bekrizadeh; Writing – Original Draft: Hakim Bekrizadeh; Writing – Review and Editing: Hakim Bekrizadeh; Visualization: Hakim Bekrizadeh; Supervision: Corresponding Author; Project Administration: Hakim Bekrizadeh; Funding Acquisition: None.

Acknowledgments

The authors received no specific funding for this research. The authors would like to thank all colleagues and reviewers whose constructive comments significantly improved the quality of this manuscript.

Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have influenced the work reported in this paper.

Artificial Intelligence Statement

Artificial intelligence (AI) tools, including large language models, were used solely for language editing and improving readability. AI tools were not used for generating ideas, performing analyses, interpreting results, or writing the scientific content. All scientific conclusions and intellectual contributions were made exclusively by the authors.

CPublisher's Note

The publisher remains neutral regarding jurisdictional claims in published maps and institutional affiliations.

References

[1] Arnold, B.C. and Nagaraja, H.N. On some properties of bivariate weighted distributions. *Communications in Statistics: Theory and Methods*, 1991, 20, 1853–1860. https://doi.org/10.1080/03610929108830602.

[2] Bairamov, I. and Kotz, S. Dependence structure and symmetry of Huang–Kotz FGM distributions and their extensions. *Metrika*, 2002, 56, 55–72. https://doi.org/10.1007/s001840200199.

[3] Bekrizadeh, H., Parham, G.A., and Zadkarmi, M.R. The New Generalization of Farlie–Gumbel–Morgenstern Copulas. *Applied Mathematical Sciences*, 2012, 71(6), 3527–3533. https://www.m-hikari.com/ams/ams-2012/ams-69-72-2012/bekrizadehAMS69-72-2012.pdf.

[4] Bekrizadeh, H., Parham, G.A., and Zadkarmi, M.R. Weighted Clayton Copulas and their Characterizations: a application to Probable Modeling of the Hydrology Data. *Journal of Data Science*, 2013, 11(2), 293–303. https://doi.org/10.6339/JDS.2013.11(2).1084.

[5] Bekrizadeh, H. Generalized Family of Copulas: Definition and Properties. *Thailand Statistician*, 2021, 19(1), 163–178. https://doi.org/10.1080/03610918.2022.2032156.

[6] Bekrizadeh, H. Generalized FGM copulas: Properties and applications. *Communications in Statistics: Simulation and Computation*, 2024, 53(2), 744–755. https://doi.org/10.1080/03610918.2022.2032156.

[7] Farlie, D.G.J. The performance of some correlation coefficients for a general bivariate distribution. *Biometrika*, 1960, 47(3–4), 307–323. https://doi.org/10.1093/biomet/47.3-4.307.

[8] Fisher, R.A. The effects of methods of ascertainment upon the estimation of frequencies. *The Annals of Eugenics*, 1934, 6, 13–25. https://doi.org/10.1111/j.1469-1809.1934.tb02105.x.

[9] Gumbel, E.J. Bivariate exponential distributions. *Journal of the American Statistical Association*, 1960, 55, 698–707. https://doi.org/10.1080/01621459.1960.10483368.

[10] Huang, J.S. and Kotz, S. Modifications of the Farlie–Gumbel–Morgenstern distributions, a tough hill to climb. *Metrika*, 1999, 49(2), 135–145. https://doi.org/10.1007/s001840050030.

[11] Joe, H. *Multivariate Models and Dependence Concepts*. London: Chapman and Hall, 1997. https://doi.org/10.1201/9780367803896.

[12] Joag-Dev, K. and Proschan, F. Negative Association of Random Variables with Applications. *Annals of Statistics*, 1983, 11, 286–295. https://doi.org/10.1214/aos/1176346079.

[13] Johnson, N.L. and Kotz, S. *Distributions in Statistics: Continuous Multivariate Distributions*. Wiley, New York, 1972. https://doi.org/10.1002/0471722065.

[14] Lai, C.D. and Xie, M. A new family of positive quadrant dependent bivariate distributions. *Statistics and Probability Letters*, 2000, 46(4), 359–364. https://doi.org/10.1016/S0167-7152(99)00122-4.

[15] Li, X. and Fang, R. A new family of bivariate copulas generated by univariate distributions. *Journal of Data Science*, 2012, 10, 1–17. https://doi.org/10.6339/JDS.2012.10(1).1011.

[16] Mahfoud, M. and Patil, G.P. On weighted distributions. In *Statistics in Theory and Practice: Essays in Honor of C.R. Rao*, G. Kallianpur et al. (eds.), North-Holland, Amsterdam, 1982, 479–492. https://doi.org/10.1002/0471667196.ess2907.pub2.

[17] Makhdoom, I. and Sakhaei, A. On dependence of the weighted Marshall-Olkin bivariate exponential model in the presence of the copula function. *Journal of Statistical Modelling: Theory and Applications*, 2023, 4(1), 157–172. https://doi.org/10.22034/jsmta.2024.21105.1125

[18] Morgenstern, D. Einfache Beispiele zweidimensionaler Verteilungen. *Mitteilungsblatt für Mathematische Statistik*, 1956, 8, 234–235. https://doi.org/10.14490/jjss.36.213.

[19] Nelsen, R.B. *An Introduction to Copulas*. Springer Series in Statistics, Springer, New York, 2006. https://doi.org/10.1007/0-387-28678-0.

[20] Pathak, A.K. and Vellaisamy, P. Various measures of dependence of a new asymmetric generalized Farlie–Gumbel–Morgenstern copulas. *Communications in Statistics: Theory and Methods*, 2015, 45(18), 5299–5317. https://doi.org/10.1080/03610926.2014.942428.

[21] Patil, G.P. and Rao, C.R. Weighted distributions and size-biased sampling with applications to wildlife populations and human families. *Biometrics*, 1978, 34, 179–189. https://doi.org/10.2307/2530008.

[22] Self, S.G. and Liang, K.Y. Asymptotic properties of maximum likelihood estimators and Likelihood ratio tests under nonstandard conditions. *Journal of the American Statistical Association*, 1987, 82(398), 605–610. https://doi.org/10.1080/01621459.1987.10478472.

[23] Rao, C.R. On discrete distributions arising out of methods of ascertainment. In G.P. Patil (Ed.), *Classical and Contagious Discrete Distributions*, Statistical Publishing Society and Pergamum Press, 1965, 320–332. https://www.jstor.org/stable/25049375.

Author(s) Bio-sketches Hakim Bekrizadeh received a Ph.D. in Statistics from Shahid Chamran University of Ahvaz. His research interests include copula, and modeling, and applied statistics. He is currently affiliated with Payame Noor University (PNU) as a faculty member. Corresponding author: Bekrizadeh@pnu.ac.ir