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Mathematical Modeling and Optimal Control of Migration-Driven Socio-Economic and Environmental Dynamics

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Abstract. This paper presents a deterministic nonlinear dynamical model to investigate the interactions among migration, unemployment, job availability, public resources, and environmental pollution. We establish that, under nonnegative initial data, the model solutions remain bounded, and we use analytical methods to demonstrate the existence of feasible endemic equilibria and to establish conditions for local and regional asymptotic stability. For the selected parameter set, four feasible endemic equilibria are identified, of which only one is locally asymptotically stable. To mitigate the adverse effects of unregulated migration surges, such as rapid resource depletion and severe pollution, we formulate an optimal control problem and apply Pontryagin's Maximum Principle to design intervention strategies that reduce migration intensity and pollution levels. Numerical simulations support the theoretical conclusions and show that a combination of socio-economic and environmental measures provides the most effective and balanced outcome. The study emphasizes the importance of coordinated policy approaches for achieving long-term regional sustainability.

Keywords. Human migration, Environmental sustainability, Unemployment dynamics, Optimal control, Socio-economic modeling.

MSC. 49J15; 34D23; 91D10.

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1 Introduction

Human migration—whether international, domestic, or rural-to-urban—has been the subject of intense scholarly debate regarding its economic, fiscal, and environmental consequences for destination regions. A comprehensive study of migration determinants identifies income and employment disparities between origin and destination as the single most consistently important driver of both internal and international migration, particularly when these disparities are accompanied by substantial unemployment or underemployment at the origin. Beyond these economic disparities, unfavorable political and institutional conditions may further encourage migration, with political freedom, legal conditions, and other socio-political factors also identified as important migration determinants [19, 20].

The arrival of migrants significantly influences labor-market dynamics in destination regions. Recent empirical studies indicate that immigration may generate adverse short-run effects on native employment, particularly among low-skilled workers, although these effects often diminish over time as labor markets adjust to immigrant inflows [7]. An agent-based modeling study for Austria that simulates a major migration shock shows that unemployment rates increase for both native-born people and former migrants, with some sectors experiencing particularly high unemployment because of competition for jobs [21]. Similarly, a panel VAR analysis of 27 EU countries demonstrates a bidirectional causal relationship between unemployment and migration, implying that each may reinforce the other over time [6]. A labor-market study of EU emigrant countries found that emigration can increase unemployment and linked this adverse effect to structural labor-market imbalances, particularly mismatches between the skills of emigrating workers and those of the workers remaining in the labor force [25]. Comparable effects are observed in the United States, where increased low-skilled immigration reduces employment rates among native youth [26].

These labor-market pressures are particularly visible in developing regions. Rapid rural-to-urban migration in India has produced high levels of informality, underemployment, and insecure work, as urban labor markets fail to generate sufficient stable employment. Across South Asia, migrants are overrepresented in low-skill and insecure occupations, reflecting strong competition for limited formal jobs [15].

Migration also affects public infrastructure and resource availability. An EU-wide fiscal study shows that migrants are net recipients of public transfers once social housing, education, and health services are taken into account, indicating increased demand for public services [4]. Local-level European studies similarly find higher expenditure on housing support, education, and welfare in regions with strong inflows [2]. In Indian megacities such as Delhi, internal migration places intense pressure on water supply, sanitation, housing, and transport infrastructure [23].

Beyond economic and infrastructural effects, migration contributes to environmental degradation. Peri-urban expansion driven by migrants has transformed settlements into polluted zones with deteriorating air and water quality [29]. Similar patterns are observed across South Asian cities, where increasing population density leads to air, water, and soil pollution owing to inadequate waste management and transport infrastructure [10, 15]. Migration-induced clustering further intensifies pressure on land and natural resources [1]. At the same time, environmental quality influences migration decisions, as cleaner cities attract more migrants while polluted regions deter movement [24]. These observations indicate a feedback relationship among migration, economic conditions, and environmental sustainability.

While the above discussion explains the phenomenon qualitatively, a deeper understanding of the interaction among migration, labor markets, resources, and environmental quality requires a rigorous analytical framework. Mathematical modeling has therefore emerged as an effective tool for analyzing such interconnected systems. Migration modeling has evolved from gravity-type formulations to nonlinear dynamical systems and spatial interaction models. Nikolova et al. [18] developed a nonlinear differential equation model describing the redistribution of population between regions on the basis of socio-economic attractiveness. More recently, Michailat [16] developed a mathematical model of migration-induced unemployment based on labor-market matching and job rationing, demonstrating how migrant inflows may increase unemployment when job opportunities are limited. Misra and Kumari [17] constructed a nonlinear unemployment model incorporating informal skill formation and job creation. Kaslik et al. [9] proposed a delay differential labor-market model and established positivity, boundedness, and global stability of equilibria. Camacho and Pérez-Barahona [3] introduced a spatial-dynamic migration model linking welfare gradients and environmental quality. In a related economic-environmental framework, Xu et al. [31] developed a stochastic model with pollution control strategies and established near-optimality conditions for reducing environmental pollution and the associated economic losses. Li et al. [14] developed a probabilistic multilateral migration framework using concepts from statistical physics, and Cosner [5] analyzed dispersal dynamics using reaction–diffusion–advection systems.

To the best of our knowledge, no existing work simultaneously couples migration pressure, labor dynamics, resource availability, and environmental pollution within a single optimal control dynamical system. Motivated by this gap, this paper develops a novel human migration model that integrates economic and environmental factors within a unified framework. The model dynamically links migration, unemployment, job availability, public resource capacity, and pollution generation through bidirectional feedback mechanisms, enabling the simultaneous analysis of labor-market competition, resource depletion, and environmental degradation. The proposed model is analyzed by establishing a bounded invariant region, determining the equilibrium points, and investigating their local and regional asymptotic stability. An optimal control strategy is incorporated to investigate policies that regulate migration inflow and

mitigate environmental deterioration, providing quantitative insight into sustainable regional management.

The remainder of the paper is organized as follows. Section 2 presents the model formulation. Section 3 introduces the notation and symbol conventions. Section 4 presents the preliminary analysis of the model. Section 5 investigates the equilibrium points and the stability behavior of the system. Section 6 presents the normalized sensitivity analysis. Section 7 is devoted to the optimal control analysis. Section 8 presents the numerical simulations, and Section 9 concludes the study with directions for future research. The derivations of the cubic equation for the endemic equilibrium and of the characteristic polynomial are given in Appendices A and B, respectively.

2 Model Formulation

We develop a deterministic nonlinear dynamical model to study the coupled interaction among migration, unemployment, job availability, public resources, and environmental pollution within a host region. The model captures the feedback structure through which migration influences socio-economic and environmental conditions, while these factors, in turn, regulate migration dynamics.

Let $M(t)$ denote the migrant population at time t , $U(t)$ the number of unemployed persons in the host region, $J(t)$ the number of available jobs, $R(t)$ the level of public resources, and $P(t)$ the pollution and waste-management pressure. Migration is assumed to increase through economic attraction and social opportunities but to be limited by infrastructural and socio-economic constraints. Simultaneously, unemployment, employment opportunities, resource availability, and pollution evolve dynamically under migration pressure and internal regulatory mechanisms. Based on these assumptions, the dimensional model is given by

$$\begin{aligned}\frac{dM}{dt} &= \alpha M \left(1 - \frac{M}{K}\right) - \beta U, \\ \frac{dU}{dt} &= \gamma M - \delta J - \eta U, \\ \frac{dJ}{dt} &= \epsilon J \left(1 - \frac{J}{L}\right) - \phi M, \\ \frac{dR}{dt} &= \lambda R \left(1 - \frac{R}{N}\right) - \theta J, \\ \frac{dP}{dt} &= \nu M - \kappa P.\end{aligned}\tag{1}$$

The first equation represents migration growth with intrinsic attraction rate α and carrying capacity K , while unemployment-driven migration outflow occurs at rate β . The second

equation describes an increase in the unemployed population through migration-induced job competition at rate γ , while available jobs reduce unemployment at rate δ . The parameter η denotes the natural unemployment decline rate associated with labor-market adjustment, informal employment, and self-employment opportunities. The third equation models job creation with intrinsic growth rate ϵ and carrying capacity L , together with migration-induced job pressure at rate ϕ . Public resources regenerate logistically at rate λ up to capacity N , while increased job availability is interpreted as a higher level of economic activity, which intensifies the utilization of public infrastructure, utilities, transportation, and municipal services and thereby accelerates public resource depletion at rate θ . Pollution accumulates proportionally to migration intensity at rate ν and is reduced through environmental mitigation at rate κ . A schematic representation of the interactions among the variables is shown in Figure 1, and the parameters are described in Table 1.

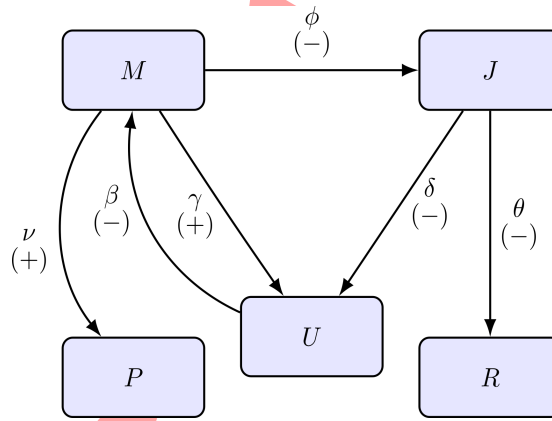


Figure 1: Schematic flowchart of the interactions among M , J , U , R , and P . Arrows indicate the direction of influence between variables.

To facilitate analytical and numerical investigation, system (1) is transformed into a nondimensional form. Since the variables evolve on different scales and have heterogeneous units, nondimensionalization reduces parameter complexity, improves numerical stability, and highlights the relative influence of competing mechanisms. We introduce the scaled variables

$$x = \frac{M}{K}, \quad u = \frac{U}{K}, \quad y = \frac{J}{L}, \quad r = \frac{R}{N}, \quad p = \frac{P}{P_0},$$

where $P_0 = \nu K / \kappa$ is the pollution scale, and define the dimensionless time variable $\tau = \alpha t$. The resulting nondimensional system is

$$\begin{aligned}
\frac{dx}{d\tau} &= x(1-x) - au, \\
\frac{du}{d\tau} &= bx - cy - du, \\
\frac{dy}{d\tau} &= ey(1-y) - fx, \\
\frac{dr}{d\tau} &= gr(1-r) - hy, \\
\frac{dp}{d\tau} &= \mu(x-p).
\end{aligned} \tag{2}$$

Here, the scaled parameters represent the relative strengths of the underlying socio-economic and environmental processes:

$$\begin{aligned}
a &= \frac{\beta}{\alpha}, \quad b = \frac{\gamma}{\alpha}, \quad c = \frac{\delta L}{\alpha K}, \quad d = \frac{\eta}{\alpha}, \\
e &= \frac{\epsilon}{\alpha}, \quad f = \frac{\phi K}{\alpha L}, \quad g = \frac{\lambda}{\alpha}, \quad h = \frac{\theta L}{\alpha N}, \quad \mu = \frac{\kappa}{\alpha}.
\end{aligned}$$

The nondimensional model preserves the qualitative dynamics of the original system while providing a mathematically convenient framework for equilibrium analysis, stability investigation, and optimal control. Consequently, all subsequent analytical and numerical results are obtained using system (2).

3 Notation and Symbol Conventions

In addition to the parameters defined in Table 1, the following notation is used in the paper.

A_0, A_1, A_2, A_3 : The coefficients of the cubic equation for the endemic equilibrium (Section 5.1).

R_1, R_2, R_3 : The Routh–Hurwitz coefficients of the characteristic polynomial (Section 5.2).

C_1, C_2, C_3, C_4, C_5 : The coefficients in the quadratic upper bound for $\dot{V}$ (Section 5.3).

a_1, a_2 : The objective-functional weights for migration and pollution, respectively (Section 7).

w_1, w_2 : The control-cost weights associated with v_1 and v_2 , respectively (Section 7).

4 Preliminary Analysis of the Model

In this section, we establish the basic mathematical properties of the nondimensional system (2). In particular, we show that solutions corresponding to nonnegative initial conditions remain bounded and socio-economically meaningful for all future time.

Table 1: Description of the parameters in the dimensional migration model (1).

Parameter	Description
α	Migration attraction rate
β	Unemployment-driven migration outflow rate
γ	Job competition rate due to migration
δ	Unemployment reduction rate due to available jobs
η	Unemployment reduction rate due to natural labor-market adjustment and self-employment
ϵ	Job creation rate
ϕ	Migration-induced job loss rate
λ	Public resource regeneration rate
θ	Public resource depletion rate associated with economic activity
ν	Pollution generation rate due to migration
κ	Pollution mitigation rate
K	Carrying capacity of migrants
L	Carrying capacity of jobs
N	Carrying capacity of public resources

4.1 Invariant Region of the Model

All state variables and parameters are assumed to be nonnegative on socio-economic grounds. We first show that the trajectories of system (2) remain uniformly bounded within a feasible region.

Lemma 1. The solution $(x(\tau), u(\tau), y(\tau), r(\tau), p(\tau))$ of system (2) with nonnegative initial conditions ultimately enters the region

$$\Omega = \{(x, u, y, r, p) \in \mathbb{R}_+^5 : 0 \leq x, y, r, p \leq 1, u \geq 0\},$$

and remains there for all sufficiently large τ .

Proof. Define

$$W(\tau) = x(\tau) + y(\tau) + r(\tau) + p(\tau).$$

Differentiating along the solutions of system (2) gives

$$\begin{aligned} \frac{dW}{d\tau} &= x(1-x) - au + ey(1-y) - fx + gr(1-r) - hy + \mu(x-p) \\ &= (1-f+\mu)x + (e-h)y + gr - \mu p - x^2 - ey^2 - gr^2 - au. \end{aligned}$$

Ignoring the nonpositive terms $-x^2$, $-ey^2$, $-gr^2$, and $-au$, we obtain

$$\frac{dW}{d\tau} \leq (1 - f + \mu)x + (e - h)y + gr - \mu p.$$

Hence, there exist positive constants Λ_W and σ such that

$$\frac{dW}{d\tau} \leq \Lambda_W - \sigma W.$$

By the comparison theorem,

$$W(\tau) \leq \frac{\Lambda_W}{\sigma} + \left(W(0) - \frac{\Lambda_W}{\sigma} \right) \exp(-\sigma\tau),$$

which implies that $x(\tau)$, $y(\tau)$, $r(\tau)$, and $p(\tau)$ remain bounded.

Now consider

$$\frac{du}{d\tau} = bx - cy - du.$$

Since $x(\tau)$ and $y(\tau)$ are bounded, there exists a constant $\Lambda_u > 0$ such that

$$\frac{du}{d\tau} \leq \Lambda_u - du.$$

Applying the comparison theorem again yields

$$u(\tau) \leq \frac{\Lambda_u}{d} + \left(u(0) - \frac{\Lambda_u}{d} \right) \exp(-d\tau),$$

which shows that $u(\tau)$ is also bounded.

Furthermore, owing to the logistic growth terms in the equations for x , y , and r , their trajectories remain confined to the interval $[0, 1]$. Consequently, from

$$\frac{dp}{d\tau} = \mu(x - p),$$

it follows that

$$-\mu p \leq \frac{dp}{d\tau} \leq \mu(1 - p).$$

Hence, by the comparison theorem, $0 \leq p(\tau) \leq 1$.

Therefore, all trajectories eventually enter and remain in the compact region Ω for all future time. Hence, system (2) is mathematically well posed, and its solutions remain socio-economically meaningful. $\square$

5 Equilibrium Points and Stability Analysis

In this section, we determine the equilibrium points of system (2) and investigate their stability properties. Particular attention is devoted to the endemic equilibrium, which represents the persistent coexistence of migration, employment, public resources, and environmental pressure.

5.1 Equilibrium Points

The equilibrium points of system (2) are obtained by setting

$$\frac{dx}{d\tau} = \frac{du}{d\tau} = \frac{dy}{d\tau} = \frac{dr}{d\tau} = \frac{dp}{d\tau} = 0.$$

Thus, the steady states $(x^*, u^*, y^*, r^*, p^*)$ satisfy

$$\begin{aligned} x(1-x) - au &= 0, \\ bx - cy - du &= 0, \\ ey(1-y) - fx &= 0, \\ gr(1-r) - hy &= 0, \\ \mu(x-p) &= 0. \end{aligned} \tag{3}$$

From the last equation of system (3),

$$p^* = x^*. \tag{4}$$

We now classify the admissible equilibria of the system.

5.1.1 Trivial Equilibrium

If $x^* = 0$, then system (3) gives $u^* = y^* = p^* = 0$ and $r^* \in \{0, 1\}$. The choice $r^* = 0$ yields the trivial equilibrium

$$E_0 = (0, 0, 0, 0, 0),$$

which represents the complete absence of migration and socio-economic activity.

5.1.2 Resource-Only Equilibrium

The remaining choice, $r^* = 1$ with $x^* = u^* = y^* = 0$, gives $p^* = 0$ by Equation (4). Thus,

$$E_1 = (0, 0, 0, 1, 0),$$

which represents a state in which migration, unemployment, employment activity, and pollution are absent, while public resources recover to their carrying capacity. Although mathematically admissible, this equilibrium does not represent a realistic socio-economic operating state.

5.1.3 Endemic Equilibrium

We now consider the coexistence equilibrium

$$E^* = (x^*, u^*, y^*, r^*, p^*),$$

in which all components are positive. From the first equation of system (3),

$$u^* = \frac{x^*(1 - x^*)}{a}. \quad (5)$$

Substituting Equation (5) into the second equation gives

$$y^* = \frac{1}{c} \left[bx^* - \frac{d}{a} x^*(1 - x^*) \right]. \quad (6)$$

Using Equation (6) in the third equation of system (3) yields the cubic equation

$$A_3 x^{*3} + A_2 x^{*2} + A_1 x^* + A_0 = 0, \quad (7)$$

whose complete algebraic derivation is provided in Appendix A. The coefficients are

$$\begin{aligned} A_3 &= -\frac{e}{c^2} \left(\frac{d}{a} \right)^2, \\ A_2 &= -\frac{2e}{c^2} \left(b - \frac{d}{a} \right) \frac{d}{a}, \\ A_1 &= \frac{e}{c} \frac{d}{a} - \frac{e}{c^2} \left(b - \frac{d}{a} \right)^2, \\ A_0 &= \frac{e}{c} \left(b - \frac{d}{a} \right) - f. \end{aligned}$$

Let

$$F(x) = A_3 x^3 + A_2 x^2 + A_1 x + A_0.$$

The following theorem establishes a sufficient condition for the existence of a feasible endemic equilibrium.

Theorem 1. Let $x_c \in (0, 1)$ be a critical point of $F(x)$, that is, $F'(x_c) = 0$. If

$$F(0)F(x_c) < 0 \quad \text{or} \quad F(x_c)F(1) < 0,$$

then system (2) admits at least one feasible endemic equilibrium in the feasible region Ω , provided that

$$b > \frac{d}{a}(1 - x^*) \quad \text{and} \quad 1 - \frac{4h}{g} y^* \geq 0.$$

Proof. Since $F(x)$ is continuous on $[0, 1]$, the Intermediate Value Theorem guarantees the existence of at least one root $x^* \in (0, 1)$ whenever

$$F(0)F(x_c) < 0 \quad \text{or} \quad F(x_c)F(1) < 0.$$

Consequently, by Equation (5),

$$u^* = \frac{x^*(1-x^*)}{a} > 0.$$

Furthermore, Equation (6) can be written as

$$y^* = \frac{x^*}{c} \left[b - \frac{d}{a}(1-x^*) \right],$$

and therefore the condition $b > \frac{d}{a}(1-x^*)$ implies $y^* > 0$. The resource component satisfies $gr^*(1-r^*) = hy^*$, whose solutions are

$$r^* = \frac{1}{2} \left(1 \pm \sqrt{1 - \frac{4h}{g} y^*} \right),$$

which are real whenever $1 - \frac{4h}{g} y^* \geq 0$. Finally, $p^* = x^* > 0$ by Equation (4). Hence, the corresponding equilibrium components define at least one feasible endemic equilibrium in Ω . $\square$

The remaining equilibrium components are therefore given by

$$\begin{aligned} u^* &= \frac{x^*(1-x^*)}{a}, \\ y^* &= \frac{1}{c} \left[bx^* - \frac{d}{a} x^*(1-x^*) \right], \\ r^* &= \frac{1}{2} \left(1 \pm \sqrt{1 - \frac{4h}{g} y^*} \right), \\ p^* &= x^*, \end{aligned}$$

and the endemic equilibrium is $E^* = (x^*, u^*, y^*, r^*, p^*)$.

The boundary equilibria E_0 and E_1 correspond to situations in which migration and socio-economic activity disappear from the system. Although mathematically admissible, these states do not represent realistic socio-economic conditions. In contrast, the endemic equilibrium describes the persistent interaction among migration, employment dynamics, public resources, and environmental pressure and thus reflects a more realistic long-term scenario. Hence, the subsequent analysis focuses on the existence and stability of the endemic equilibrium.

Remark 1. Uniqueness of the endemic equilibrium is not claimed in general. For the parameter values in Table 3, the model yields multiple endemic equilibria, whose feasibility and individual stability are analyzed in Section 8.

5.2 Local Stability Analysis of the Endemic Equilibrium Point

The local behavior of system (2) near an equilibrium point is determined by the eigenvalues of the corresponding Jacobian matrix. Let $E^* = (x^*, u^*, y^*, r^*, p^*)$ be the endemic equilibrium. The Jacobian matrix evaluated at E^* is

$$\mathbf{J}(E^*) = \begin{pmatrix} 1 - 2x^* & -a & 0 & 0 & 0 \\ b & -d & -c & 0 & 0 \\ -f & 0 & e(1 - 2y^*) & 0 & 0 \\ 0 & 0 & -h & g(1 - 2r^*) & 0 \\ \mu & 0 & 0 & 0 & -\mu \end{pmatrix}. \quad (8)$$

The matrix (8) has a block lower-triangular structure, so the eigenvalues of the full system are obtained from the socio-economic subsystem (x, u, y) together with

$$\lambda_4 = g(1 - 2r^*), \quad \lambda_5 = -\mu.$$

Since $\mu > 0$, the pollution component is always locally stable, while the resource dynamics are locally stable whenever $r^* > \frac{1}{2}$.

The characteristic equation of the subsystem (x, u, y) is

$$\lambda^3 + R_1\lambda^2 + R_2\lambda + R_3 = 0, \quad (9)$$

where

$$\begin{aligned} R_1 &= d - (1 - 2x^*) - e(1 - 2y^*), \\ R_2 &= -d(1 - 2x^*) - de(1 - 2y^*) + (1 - 2x^*)e(1 - 2y^*) + ab, \\ R_3 &= (1 - 2x^*)de(1 - 2y^*) - abe(1 - 2y^*) + acf. \end{aligned}$$

The derivation of the characteristic polynomial and of its coefficients is given in Appendix B.

According to the Routh–Hurwitz criterion [11], all eigenvalues of Equation (9) have negative real parts if and only if

$$R_1 > 0, \quad R_2 > 0, \quad R_3 > 0, \quad R_1R_2 > R_3.$$

Hence, we obtain the following result.

Theorem 2. An endemic equilibrium $E^* = (x^*, u^*, y^*, r^*, p^*)$ of system (2) is locally asymptotically stable if

$$r^* > \frac{1}{2}, \quad R_1 > 0, \quad R_2 > 0, \quad R_3 > 0, \quad R_1R_2 > R_3,$$

where R_1, R_2 , and R_3 are the coefficients of Equation (9).

5.3 Regional Asymptotic Stability Analysis

Let

$$V(x, u, y, r, p) = \frac{1}{2} \left[(x - x^*)^2 + (u - u^*)^2 + (y - y^*)^2 + (r - r^*)^2 + (p - p^*)^2 \right].$$

For $\varepsilon > 0$, define the Lyapunov sublevel set [12, 27, 28, 30]

$$\mathcal{D}_\varepsilon = \{ (x, u, y, r, p) \in \Omega : V(x, u, y, r, p) \leq \varepsilon \}.$$

Here, ε is chosen sufficiently small that $\mathcal{D}_\varepsilon \subseteq \Omega$ and E^* is the only equilibrium contained in $\mathcal{D}_\varepsilon$. Thus, $E^* \in \text{int}(\mathcal{D}_\varepsilon)$.

Theorem 3. Let $E^* = (x^*, u^*, y^*, r^*, p^*)$ be an endemic equilibrium of system (2). For $\varepsilon > 0$, let $\mathcal{D}_\varepsilon$ be defined as above and set $\delta_\varepsilon = \sqrt{2\varepsilon}$. Suppose that

$$\begin{aligned} 2x^* - 1 - \delta_\varepsilon - \frac{|b-a|}{2} - \frac{f}{2} - \frac{\mu}{2} &> 0, \\ d - \frac{|b-a|}{2} - \frac{c}{2} &> 0, \\ e(2y^* - 1) - e\delta_\varepsilon - \frac{c}{2} - \frac{f}{2} - \frac{h}{2} &> 0, \\ g(2r^* - 1) - g\delta_\varepsilon - \frac{h}{2} &> 0, \quad \mu > 0. \end{aligned}$$

Then E^* is asymptotically stable with respect to $\mathcal{D}_\varepsilon$; in particular, every solution with initial condition in $\mathcal{D}_\varepsilon$ converges to E^* as $\tau \rightarrow \infty$.

Proof. Define the deviation variables

$$X = x - x^*, \quad Y = u - u^*, \quad Z = y - y^*, \quad W = r - r^*, \quad Q = p - p^*,$$

and consider the Lyapunov function

$$V = \frac{1}{2} (X^2 + Y^2 + Z^2 + W^2 + Q^2),$$

which is positive definite with respect to E^* . Differentiating along the solutions of system (2), we obtain

$$\begin{aligned} \dot{V} = & X[x(1-x) - x^*(1-x^*) - aY] + Y[bX - cZ - dY] \\ & + Z[e(y(1-y) - y^*(1-y^*)) - fX] \\ & + W[g(r(1-r) - r^*(1-r^*)) - hZ] + \mu Q(X - Q). \end{aligned}$$

Using the identities

$$\begin{aligned}x(1-x) - x^*(1-x^*) &= X(1-x-x^*), \\y(1-y) - y^*(1-y^*) &= Z(1-y-y^*), \\r(1-r) - r^*(1-r^*) &= W(1-r-r^*),\end{aligned}$$

we have

$$\begin{aligned}\dot{V} &= (1-x-x^*)X^2 + (b-a)XY - cYZ - dY^2 \\&\quad + e(1-y-y^*)Z^2 - fXZ + g(1-r-r^*)W^2 - hZW \\&\quad + \mu XQ - \mu Q^2.\end{aligned}$$

For $(x, u, y, r, p) \in \mathcal{D}_\varepsilon$,

$$|X|, |Y|, |Z|, |W|, |Q| \leq \sqrt{2\varepsilon} = \delta_\varepsilon.$$

Hence, $x \geq x^* - \delta_\varepsilon$, $y \geq y^* - \delta_\varepsilon$, and $r \geq r^* - \delta_\varepsilon$, and therefore

$$\begin{aligned}1-x-x^* &\leq 1-2x^* + \delta_\varepsilon, \\e(1-y-y^*) &\leq e(1-2y^*) + e\delta_\varepsilon, \\g(1-r-r^*) &\leq g(1-2r^*) + g\delta_\varepsilon.\end{aligned}$$

Consequently,

$$\begin{aligned}\dot{V} &\leq (1-2x^* + \delta_\varepsilon)X^2 + [e(1-2y^*) + e\delta_\varepsilon]Z^2 \\&\quad + [g(1-2r^*) + g\delta_\varepsilon]W^2 - dY^2 - \mu Q^2 \\&\quad + (b-a)XY - cYZ - fXZ - hZW + \mu XQ.\end{aligned}$$

Applying Young's inequality $|st| \leq \frac{1}{2}(s^2 + t^2)$ to the mixed terms gives

$$\dot{V} \leq -C_1(\varepsilon)X^2 - C_2Y^2 - C_3(\varepsilon)Z^2 - C_4(\varepsilon)W^2 - C_5Q^2,$$

where

$$\begin{aligned}C_1(\varepsilon) &= 2x^* - 1 - \delta_\varepsilon - \frac{|b-a|}{2} - \frac{f}{2} - \frac{\mu}{2}, \\C_2 &= d - \frac{|b-a|}{2} - \frac{c}{2}, \\C_3(\varepsilon) &= e(2y^* - 1) - e\delta_\varepsilon - \frac{c}{2} - \frac{f}{2} - \frac{h}{2}, \\C_4(\varepsilon) &= g(2r^* - 1) - g\delta_\varepsilon - \frac{h}{2}, \\C_5 &= \frac{\mu}{2}.\end{aligned}$$

By hypothesis, $C_1(\varepsilon)$, C_2 , $C_3(\varepsilon)$, $C_4(\varepsilon)$, and C_5 are all positive. Let

$$k = \min\{C_1(\varepsilon), C_2, C_3(\varepsilon), C_4(\varepsilon), C_5\} > 0.$$

Then

$$\dot{V} \leq -k(X^2 + Y^2 + Z^2 + W^2 + Q^2) = -2kV.$$

In particular, on the boundary $V = \varepsilon$ we have $\dot{V} \leq -2k\varepsilon < 0$, so $\mathcal{D}_\varepsilon$ is positively invariant. Moreover, by Gronwall's inequality,

$$V(\tau) \leq V(0) \exp(-2k\tau).$$

Since $V = \frac{1}{2}(X^2 + Y^2 + Z^2 + W^2 + Q^2)$, it follows that

$$\|(X, Y, Z, W, Q)\|_2 \leq \sqrt{2V(0)} \exp(-k\tau) \rightarrow 0 \quad \text{as } \tau \rightarrow \infty.$$

Hence, $(X, Y, Z, W, Q) \rightarrow (0, 0, 0, 0, 0)$, that is, $(x, u, y, r, p) \rightarrow (x^*, u^*, y^*, r^*, p^*)$. Thus, every solution originating in $\mathcal{D}_\varepsilon$ converges to E^* , and E^* is asymptotically stable with respect to $\mathcal{D}_\varepsilon$. $\square$

Remark 2. Let

$$\mathcal{B}(E^*) = \left\{ X_0 \in \Omega : \lim_{\tau \rightarrow \infty} \phi(\tau, X_0) = E^* \right\}$$

denote the basin of attraction of E^* , where $\phi(\tau, X_0)$ is the solution of system (2) starting at X_0 . Since other endemic equilibria may exist in Ω , the stability result is formulated within the Lyapunov sublevel set $\mathcal{D}_\varepsilon$. Theorem 3 provides the inclusion $\mathcal{D}_\varepsilon \subseteq \mathcal{B}(E^*)$. Thus, $\mathcal{D}_\varepsilon$ is a positively invariant, analytically certified subset of the basin of attraction.

6 Normalized Sensitivity Analysis

To quantify the relative influence of selected model parameters on the stable endemic equilibrium, a normalized sensitivity analysis was performed using the parameter values given in Table 3. The analysis was evaluated at $E_1^* = (0.891076, 0.485298, 0.818571, 0.728941, 0.891076)$.

For an equilibrium component q^* and a parameter ρ , the normalized sensitivity index is defined by

$$S_\rho^{q^*} = \frac{\partial q^*}{\partial \rho} \frac{\rho}{q^*}.$$

The normalized sensitivity indices with respect to a, b, c, f , and h are reported in Table 2.

The results in Table 2 show that the strongest relative effect on the equilibrium unemployment level is exerted by b , with $S_b^{u^*} = 0.984491$, and that the dominant impact on the resource level is exerted by h , with $S_h^{r^*} = -0.591985$. Among the selected parameters, f has the greatest influence on job availability, with $S_f^{y^*} = -0.283325$. The remaining indices are smaller in magnitude, which implies relatively weak effects on the corresponding equilibrium components.

Table 2: Normalized sensitivity indices of the stable endemic equilibrium E_1^* with respect to the selected model parameters.

Parameter	x^*	u^*	y^*	r^*	p^*
a	-0.119470	-0.142123	0.034019	-0.020139	-0.119470
b	-0.137102	0.984491	0.039041	-0.023111	-0.137102
c	0.017632	-0.126614	-0.005021	0.002972	0.017632
f	-0.005021	0.036054	-0.283325	0.167724	-0.005021
h	0	0	0	-0.591985	0

7 Optimal Control Analysis

In this section, system (2) is extended into an optimal control framework to investigate intervention strategies aimed at reducing migration pressure and environmental pollution. Two bounded control functions are introduced. The control $v_1(\tau)$ represents migration regulation measures, such as entry restrictions, relocation policies, settlement quotas, or controlled admission schemes, while $v_2(\tau)$ represents pollution mitigation efforts, including waste management and environmental regulation. Incorporating these controls into the nondimensional model gives

$$\begin{aligned}
 \frac{dx}{d\tau} &= x(1-x) - au - v_1(\tau)x, \\
 \frac{du}{d\tau} &= bx - cy - du, \\
 \frac{dy}{d\tau} &= ey(1-y) - fx, \\
 \frac{dr}{d\tau} &= gr(1-r) - hy, \\
 \frac{dp}{d\tau} &= \mu(x-p) - v_2(\tau)p,
 \end{aligned} \tag{10}$$

subject to the initial conditions

$$x(0) = x_0, \quad u(0) = u_0, \quad y(0) = y_0, \quad r(0) = r_0, \quad p(0) = p_0,$$

where all initial values are nonnegative. The admissible control set is defined as

$$\mathcal{U} = \left\{ (v_1, v_2) : 0 \leq v_1(\tau) \leq v_{1,\max}, 0 \leq v_2(\tau) \leq v_{2,\max}, \tau \in [0, T] \right\}.$$

The objective is to reduce migration pressure and pollution while minimizing the implementation costs of the control measures. This is represented by the performance functional

$$J(v_1, v_2) = \int_0^T \left[a_1 x(\tau) + a_2 p(\tau) + \frac{1}{2} w_1 v_1^2(\tau) + \frac{1}{2} w_2 v_2^2(\tau) \right] d\tau, \tag{11}$$

where a_1 and a_2 measure the relative importance of reducing migration and pollution, while w_1 and w_2 denote the costs associated with migration regulation and pollution mitigation, respectively. The optimal control problem is to determine $(v_1^*, v_2^*) \in \mathcal{U}$ such that

$$J(v_1^*, v_2^*) = \min_{(v_1, v_2) \in \mathcal{U}} J(v_1, v_2),$$

subject to the controlled system (10).

The selected controls act in a complementary manner: migration regulation reduces migration pressure directly, while pollution mitigation accelerates environmental recovery. Consequently, the optimal control framework provides a quantitative basis for evaluating sustainable policy interventions for migration and environmental management.

7.1 Existence of Optimal Control

Before deriving the optimality conditions, we establish the existence of an optimal control pair using the Fleming–Rishel existence theorem [8].

Theorem 4. Consider the optimal control problem governed by system (10) with the objective functional (11). Then there exists an optimal control pair $\mathbf{v}^*(\tau) = (v_1^*(\tau), v_2^*(\tau)) \in \mathcal{U}$ such that

$$J(v_1^*, v_2^*) = \min_{(v_1, v_2) \in \mathcal{U}} J(v_1, v_2).$$

Proof. The admissible control set $\mathcal{U}$ is nonempty, closed, bounded, and convex. The right-hand side of system (10) is continuous in both the state and control variables and is locally Lipschitz continuous with respect to the state variables in the positively invariant region Ω . Hence, for every admissible control pair, there exists a unique state solution on $[0, T]$.

The integrand of the objective functional is

$$L = a_1 x + a_2 p + \frac{1}{2} w_1 v_1^2 + \frac{1}{2} w_2 v_2^2.$$

Since $w_1, w_2 > 0$, the integrand is convex in the control variables (v_1, v_2) . Moreover,

$$L \geq \frac{1}{2} \min(w_1, w_2) (v_1^2 + v_2^2) - (a_1 + a_2),$$

which satisfies the required coercivity condition. Therefore, all conditions of the Fleming–Rishel existence theorem are satisfied, and hence there exists an optimal control pair (v_1^*, v_2^*) minimizing the functional J . $\square$

7.2 Characterization of Optimal Control

We now derive the necessary optimality conditions using Pontryagin's Maximum Principle [13, 22].

Theorem 5. Let (v_1^*, v_2^*) be the optimal controls minimizing the objective functional (11) subject to system (10), and let $(x^*, u^*, y^*, r^*, p^*)$ be the corresponding optimal state trajectories. Then there exist adjoint variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ satisfying

$$\begin{aligned}\frac{d\lambda_1}{d\tau} &= -a_1 - \lambda_1(1 - 2x^* - v_1^*) - b\lambda_2 + f\lambda_3 - \mu\lambda_5, \\ \frac{d\lambda_2}{d\tau} &= a\lambda_1 + d\lambda_2, \\ \frac{d\lambda_3}{d\tau} &= c\lambda_2 - e(1 - 2y^*)\lambda_3 + h\lambda_4, \\ \frac{d\lambda_4}{d\tau} &= -g(1 - 2r^*)\lambda_4, \\ \frac{d\lambda_5}{d\tau} &= -a_2 + (\mu + v_2^*)\lambda_5,\end{aligned}\tag{12}$$

with the transversality conditions

$$\lambda_i(T) = 0, \quad i = 1, \dots, 5.$$

Furthermore, the optimal controls are characterized by

$$v_1^*(\tau) = \min \left\{ v_{1,\max}, \max \left(0, \frac{\lambda_1(\tau) x^*(\tau)}{w_1} \right) \right\} \tag{13}$$

and

$$v_2^*(\tau) = \min \left\{ v_{2,\max}, \max \left(0, \frac{\lambda_5(\tau) p^*(\tau)}{w_2} \right) \right\}. \tag{14}$$

Proof. The Hamiltonian associated with system (10) is

$$\begin{aligned}H &= a_1 x + a_2 p + \frac{1}{2} w_1 v_1^2 + \frac{1}{2} w_2 v_2^2 \\ &\quad + \lambda_1 [x(1 - x) - au - v_1 x] + \lambda_2 [bx - cy - du] \\ &\quad + \lambda_3 [ey(1 - y) - fx] + \lambda_4 [gr(1 - r) - hy] \\ &\quad + \lambda_5 [\mu(x - p) - v_2 p].\end{aligned}$$

By Pontryagin's Maximum Principle,

$$\frac{d\lambda_i}{d\tau} = -\frac{\partial H}{\partial X_i}, \quad X_i \in \{x, u, y, r, p\},$$

which yields the adjoint system (12) together with the transversality conditions $\lambda_i(T) = 0$, $i = 1, \dots, 5$.

Since the optimal controls minimize the Hamiltonian pointwise, they satisfy the stationarity conditions

$$\frac{\partial H}{\partial v_1} = 0, \quad \frac{\partial H}{\partial v_2} = 0.$$

Furthermore,

$$\frac{\partial^2 H}{\partial v_1^2} = w_1 > 0, \quad \frac{\partial^2 H}{\partial v_2^2} = w_2 > 0,$$

so the Hamiltonian is strictly convex with respect to the control variables. Therefore, the stationary controls obtained from $\partial H / \partial v_i = 0$ ($i = 1, 2$) are the unique pointwise minimizers of the Hamiltonian, namely

$$v_1 = \frac{\lambda_1 x}{w_1}, \quad v_2 = \frac{\lambda_5 p}{w_2}.$$

Using the admissible bounds $0 \leq v_1(\tau) \leq v_{1,\max}$ and $0 \leq v_2(\tau) \leq v_{2,\max}$, the optimal controls are obtained by projection onto the admissible interval, which yields Equations (13) and (14). Therefore, the optimality system consists of the state equations, the adjoint equations, and the above control characterizations. The resulting optimality system was solved numerically using the forward–backward sweep method, as described in Section 8: the state equations were integrated forward in time, the adjoint equations were integrated backward from the terminal conditions, and the controls were updated iteratively until convergence. $\square$

8 Numerical Simulation

To illustrate the analytical results and explore the dynamics of the migration–environment system, numerical simulations were carried out in MATLAB with the parameter set given in Table 3. Since the primary objective of this work is theoretical analysis, the parameter values were selected to demonstrate the qualitative behavior of the proposed model rather than to reproduce a specific real-world migration scenario.

For the parameter values listed in Table 3, the cubic equation (7) has three real roots, of which two are feasible, namely $x^* = 0.891076$ and $x^* = 0.878549$, while the third root is negative and therefore infeasible. Each feasible value of x^* yields two admissible values of r^* from the corresponding quadratic equation for r^* (Section 5.1.3), resulting in four feasible endemic equilibrium points:

$$E_1^* = (0.891076, 0.485298, 0.818571, 0.728941, 0.891076),$$

$$E_2^* = (0.891076, 0.485298, 0.818571, 0.271059, 0.891076),$$

$$E_3^* = (0.878549, 0.533503, 0.178169, 0.954966, 0.878549),$$

$$E_4^* = (0.878549, 0.533503, 0.178169, 0.045034, 0.878549).$$

Table 3: Parameter values used in the numerical simulations: (a) dimensional parameters and their units, and (b) the corresponding nondimensional parameters obtained from the dimensional parameters in (a). All values are assumed within reasonable ranges to illustrate the qualitative behavior of the model and are not calibrated using empirical data.

Parameter	Value	Unit
<i>(a) Dimensional parameters</i>		
α	0.035	day ⁻¹
β	0.007	day ⁻¹
γ	0.00175	day ⁻¹
δ	0.00049	person/(job · day)
η	0.0028	day ⁻¹
ϵ	0.0084	day ⁻¹
ϕ	0.0007	job/(person · day)
λ	0.01015	day ⁻¹
θ	4.9×10^{-6}	resource index/(job · day)
ν	1.295×10^{-5}	pollution pressure index/(person · day)
κ	0.01295	day ⁻¹
K	100 000	person
L	50 000	job
N	100	resource index
P_0	100	pollution pressure index
<i>(b) Nondimensional parameters</i>		
a	0.20	dimensionless
b	0.05	dimensionless
c	0.007	dimensionless
d	0.08	dimensionless
e	0.24	dimensionless
f	0.04	dimensionless
g	0.29	dimensionless
h	0.07	dimensionless
μ	0.37	dimensionless

The local stability of these equilibria was examined using the eigenvalues of the full Jacobian matrix, together with the Routh–Hurwitz criteria for the (x, u, y) subsystem and the condition $r^* > 1/2$ for stability of the resource component (Theorem 2).

For E_1^* , the five eigenvalues are -0.132786 , -0.370000 , -0.151313 , -0.096008 , and -0.767744 , so all eigenvalues have negative real parts. Moreover, $r^* = 0.728941 > 1/2$, and the corresponding Routh–Hurwitz quantities are $R_1 = 1.015066$, $R_2 = 0.204407$, $R_3 = 0.011153$, and $R_1 R_2 - R_3 = 0.196334 > 0$. Hence, $R_1 > 0$, $R_2 > 0$, $R_3 > 0$, and $R_1 R_2 > R_3$, together with $r^* > 1/2$, are all satisfied, confirming that E_1^* is locally asymptotically stable.

For E_2^* , the five eigenvalues are 0.132786 , -0.370000 , -0.151313 , -0.096008 , and -0.767744 , indicating instability. Moreover, $r^* = 0.271059 < 1/2$, and the corresponding resource eigenvalue is $\lambda_4 = g(1 - 2r^*) = 0.132786 > 0$. Thus, the required condition $r^* > 1/2$ is violated, and E_2^* is unstable.

For E_3^* , the five eigenvalues are -0.263880 , -0.370000 , 0.154228 , -0.094759 , and -0.742089 , so E_3^* is unstable. Here, $r^* = 0.954966 > 1/2$, so the resource stability condition is satisfied; however, the Routh–Hurwitz quantities are $R_1 = 0.682619$, $R_2 = -0.058746$, $R_3 = -0.010845$, and $R_1 R_2 - R_3 = -0.029256 < 0$. Thus, the conditions $R_2 > 0$, $R_3 > 0$, and $R_1 R_2 > R_3$ are violated, and the Routh–Hurwitz criterion is not satisfied.

For E_4^* , the five eigenvalues are 0.263880 , -0.370000 , 0.154228 , -0.094759 , and -0.742089 , indicating instability. Moreover, $r^* = 0.045034 < 1/2$, giving the resource eigenvalue $\lambda_4 = g(1 - 2r^*) = 0.263880 > 0$. Hence, the required condition $r^* > 1/2$ is violated, which is sufficient to establish the instability of E_4^* . Therefore, among the four feasible endemic equilibria, only E_1^* is locally asymptotically stable.

Consequently, the stable endemic equilibrium

$$(x^*, u^*, y^*, r^*, p^*) = (0.891076, 0.485298, 0.818571, 0.728941, 0.891076)$$

was selected for further analysis. The simulations were initiated from

$$(x_0, u_0, y_0, r_0, p_0) = (0.15, 0.05, 0.40, 0.80, 0.20),$$

representing low migration and unemployment levels together with moderate employment opportunities and relatively high resource availability. The dimensionless state variables x , u , y , r , and p represent the normalized levels of migration, unemployment, job availability, public resource availability, and pollution, respectively, so their numerical values describe relative levels rather than dimensional magnitudes.

The uncontrolled dynamics of the system are illustrated in Figure 2. Initially, migration increases rapidly as the host region attracts incoming population, accompanied by a gradual rise in unemployment and pollution. Job availability also increases as the regional economy adjusts to the changing population and labor-market conditions, while the public resource level initially rises before gradually declining as sustained economic activity increases the utilization of

public infrastructure, utilities, transportation, and other shared resources. Thus, in the absence of intervention, the host region experiences a rapid increase in migration and pollution pressure, together with increased unemployment, while employment opportunities and available public resources adjust to sustained socio-economic demand. As time progresses, the internal feedback mechanisms of the system regulate these changes, and all state variables gradually converge toward the selected endemic equilibrium E_1^* .

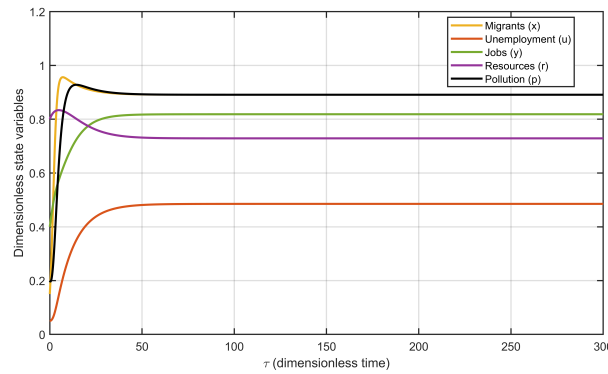


Figure 2: Time evolution of the nondimensional state variables showing convergence of the trajectories toward the endemic equilibrium E_1^* .

For a physical interpretation of the dimensionless simulation time, we assume the migration attraction rate to be $\alpha = 0.035 \text{ day}^{-1}$. Since $\tau = \alpha t$, this time scale implies that one unit of dimensionless time corresponds to approximately 28.57 days in real time. Accordingly, the simulation interval $0 \leq \tau \leq 300$ represents approximately 8571 days, or 23.5 years, and may therefore be interpreted as the long-term socio-economic and environmental evolution of the host region. The smooth convergence without sustained oscillations is consistent with the asymptotic stability of E_1^* established within the corresponding stability domain.

For the numerical parameter set, we choose $\varepsilon = 0.005$, which gives $\delta_\varepsilon = \sqrt{2\varepsilon} = 0.1$. The coefficients in the quadratic upper bound for the Lyapunov derivative (Theorem 3) are then $C_1(\varepsilon) = 0.402152$, $C_3(\varepsilon) = 0.070414$, $C_4(\varepsilon) = 0.068786$, $C_5 = 0.185$, and

$$C_2 = 0.08 - \frac{|0.05 - 0.20|}{2} - \frac{0.007}{2} = 0.0015,$$

which gives $k = \min\{C_1(\varepsilon), C_2, C_3(\varepsilon), C_4(\varepsilon), C_5\} = 0.0015 > 0$. Hence, the Lyapunov estimate becomes $\dot{V} \leq -2kV = -0.003V$, and therefore $V(\tau) \leq V(0) \exp(-0.003\tau)$. Thus, the sufficient conditions of the regional asymptotic stability result (Theorem 3) are numerically satisfied for the selected equilibrium E_1^* within the chosen stability domain $\mathcal{D}_{0.005}$. This behavior is further supported by the two-dimensional and three-dimensional phase portraits shown in Figures 3 and 4, where trajectories initiated within the corresponding stability domain move inward toward E_1^* . In physical terms, this indicates that different initial levels of migration, un-

employment, employment opportunities, public resources, and pollution converge toward the same long-term socio-economic and environmental state within the stability domain of E_1^* .

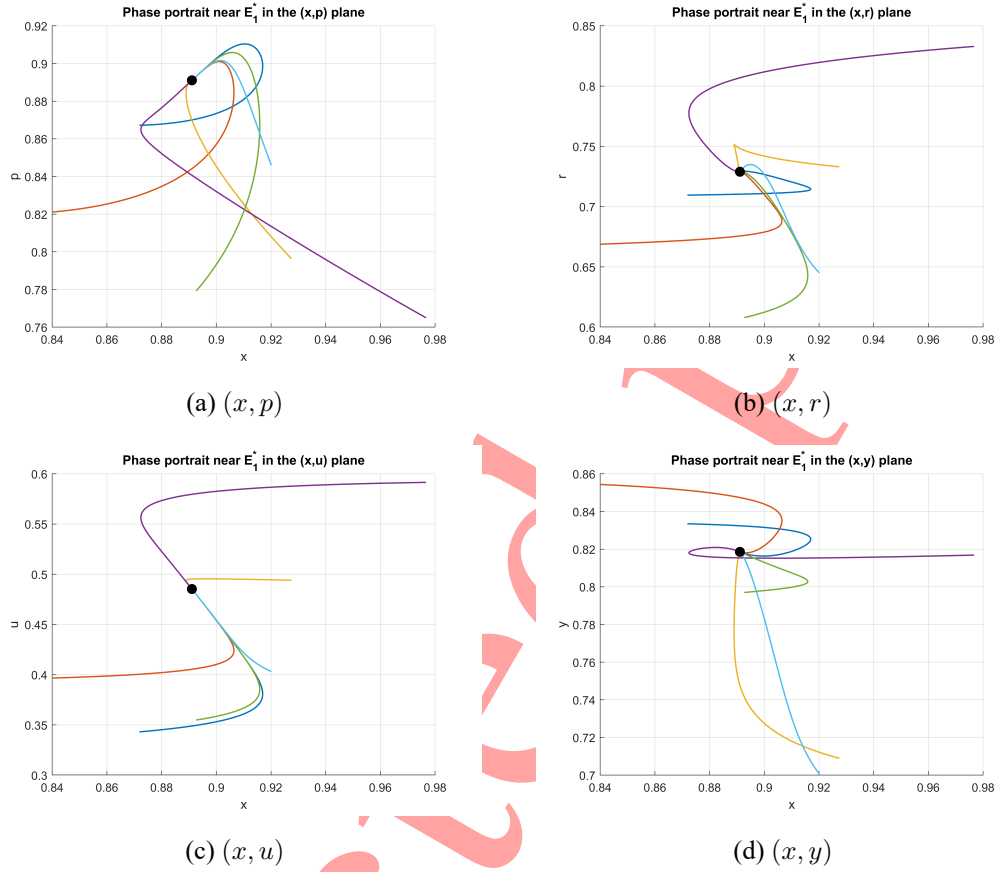


Figure 3: Two-dimensional phase portraits near the stable endemic equilibrium E_1^* , showing the convergence of nearby trajectories to the equilibrium.

The local behavior of the remaining feasible endemic equilibria is illustrated by the phase portraits shown in Figures 5, 6, and 7. For E_2^* , the nearby trajectories exhibit local saddle-type behavior associated with the unstable resource direction. For E_3^* , the trajectories depart from the equilibrium in accordance with the instability of the (x, u, y) subsystem. For E_4^* , the departure of nearby trajectories is consistent with its instability in both the resource direction and the (x, u, y) subsystem. These phase portraits provide qualitative corroboration of the instability established by the corresponding eigenvalue and Routh–Hurwitz analyses.

To illustrate the effect of the migration-related parameter $a = \beta/\alpha$, numerical trajectories were examined under perturbations of its baseline value. Figure 8 presents the resulting time evolution of the state variables for variations of up to $\pm 30\%$ in a . Increasing a represents a stronger unemployment-driven migration outflow relative to the underlying migration attraction. Consequently, higher values of a reduce the long-term migration level and the asso-

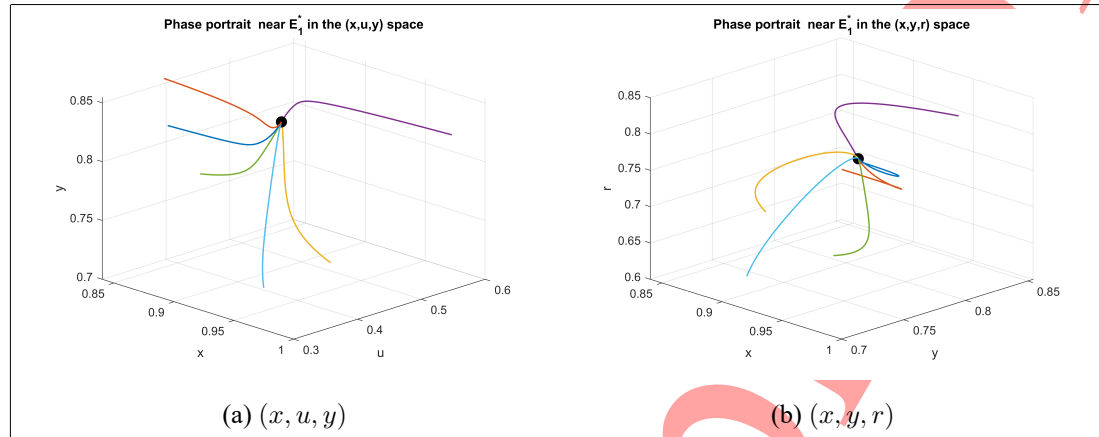


Figure 4: Three-dimensional phase portraits near the stable endemic equilibrium E_1^* , showing the convergence of nearby trajectories to the equilibrium.

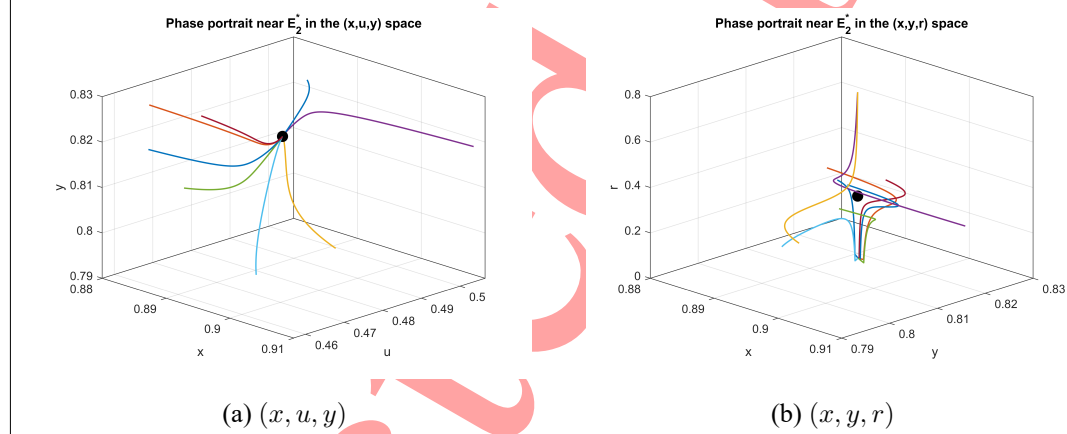


Figure 5: Phase portraits near the unstable endemic equilibrium E_2^* , showing the local saddle-type dynamics of nearby trajectories.

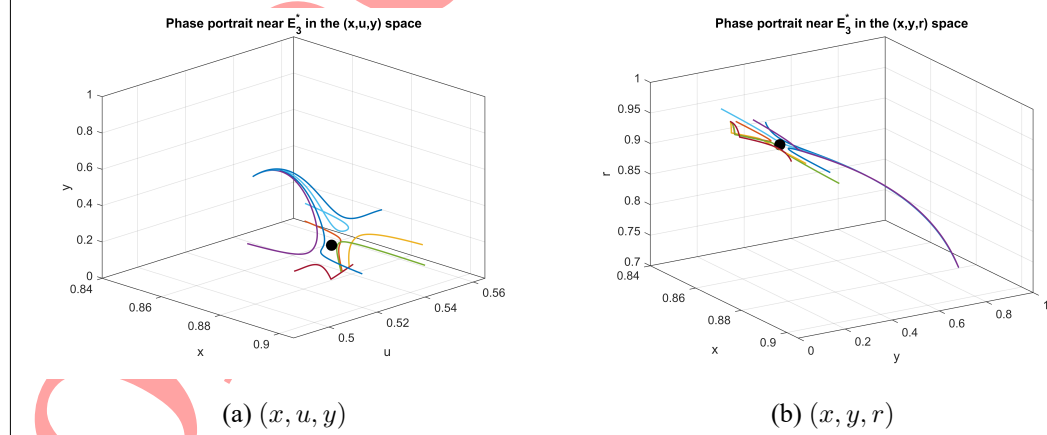


Figure 6: Phase portraits near the unstable endemic equilibrium E_3^* , showing the departure of nearby trajectories from the equilibrium.

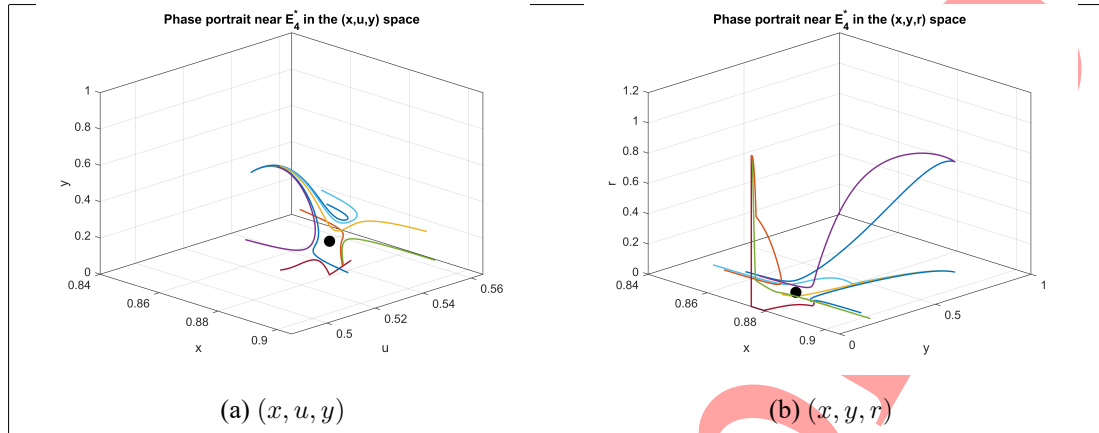


Figure 7: Phase portraits near the unstable endemic equilibrium E_4^* , showing the departure of nearby trajectories from the equilibrium.

ciated pollution pressure, while also producing corresponding adjustments in unemployment, job availability, and public resource utilization. These variations remain bounded and converge to nearby steady states, indicating that the qualitative long-term behavior of the system is preserved under moderate changes in the migration-related parameter.

Motivated by these observations, optimal control strategies were implemented to regulate migration and reduce environmental degradation. For the numerical simulations, the weighting constants were chosen as $a_1 = 1$, $a_2 = 1$, $w_1 = 20$, and $w_2 = 70$. The optimality system was solved using a forward–backward sweep procedure. At the n -th iteration, the current control profiles $v_1^{(n)}(\tau)$ and $v_2^{(n)}(\tau)$ were used to integrate the state equations forward in time, after which the resulting state trajectories were used to integrate the adjoint equations backward from the prescribed terminal conditions. The updated controls were then computed pointwise from the optimality conditions (13) and (14), with upper bounds $v_{1,\max} = v_{2,\max} = 1$, as

$$v_1^{(n+1)}(\tau) = \min \left\{ 1, \max \left\{ 0, \frac{\lambda_1^{(n)}(\tau) x^{(n)}(\tau)}{w_1} \right\} \right\},$$

$$v_2^{(n+1)}(\tau) = \min \left\{ 1, \max \left\{ 0, \frac{\lambda_5^{(n)}(\tau) p^{(n)}(\tau)}{w_2} \right\} \right\},$$

with the inactive control fixed at zero in the corresponding single-control strategy. The state and adjoint equations were integrated using the classical fourth-order Runge–Kutta method. The control interval $0 \leq \tau \leq 10$ was divided into 1000 uniform subintervals, giving a step size $\Delta\tau = 0.01$. The iterations were continued until the maximum absolute difference between successive control profiles was less than 10^{-6} , with a maximum of 100 iterations. The same numerical procedure and convergence criterion were used for all three control strategies.

Three policy scenarios were investigated. Figure 9 shows Strategy A, in which only migration control is applied. This strategy reduces migration pressure, lowers unemployment and

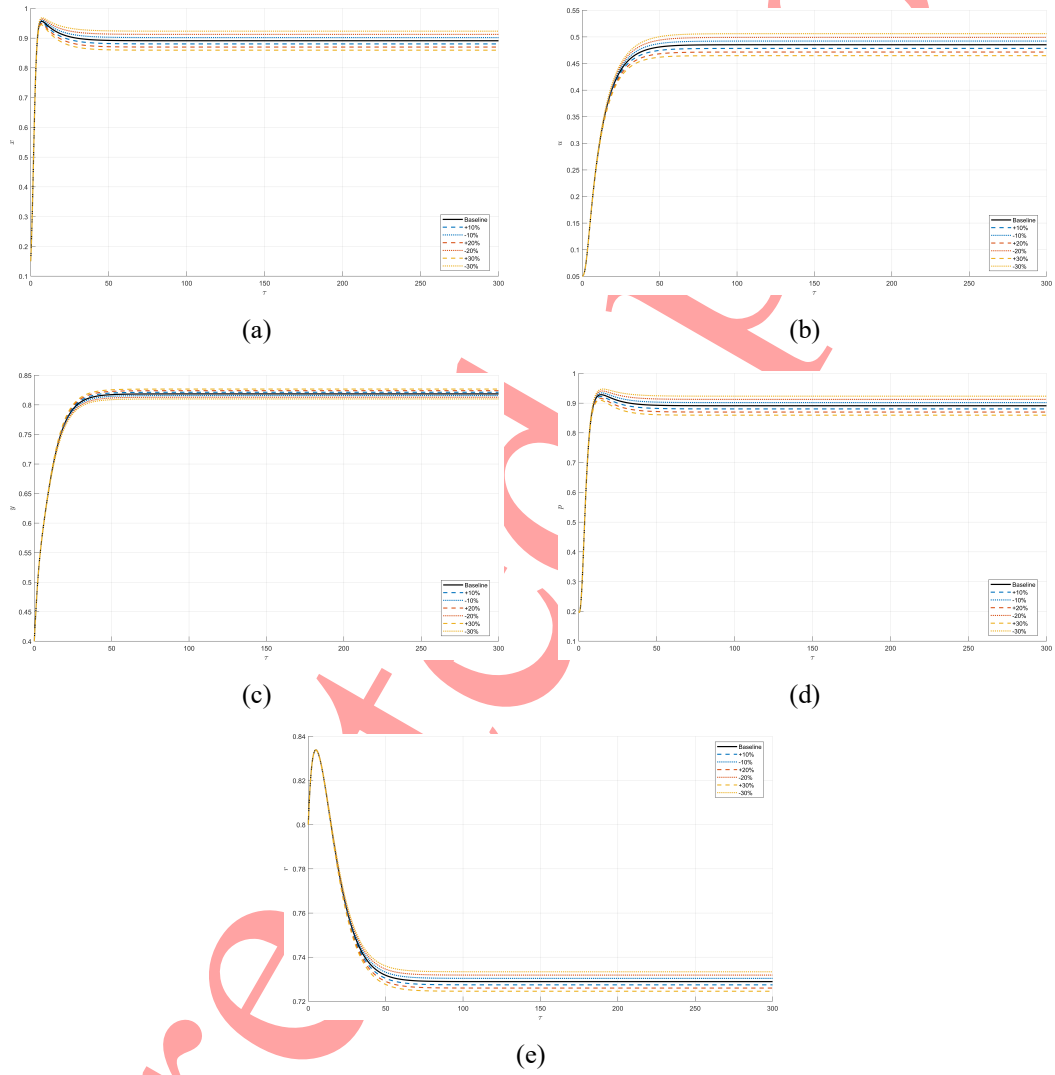


Figure 8: Time evolution of the state variables under $\pm 10\%$, $\pm 20\%$, and $\pm 30\%$ perturbations in the migration-related parameter a . The Panels (a)–(e) show, respectively, migration $x(\tau)$, unemployment $u(\tau)$, job availability $y(\tau)$, pollution $p(\tau)$ and public resource availability $r(\tau)$. The curves represent the baseline value of a and its $\pm 10\%$, $\pm 20\%$, and $\pm 30\%$ perturbations.

pollution, and promotes higher job availability. The resource level under control is somewhat lower than in the uncontrolled situation because improved working conditions and greater economic participation increase the use of public infrastructure and shared resources.

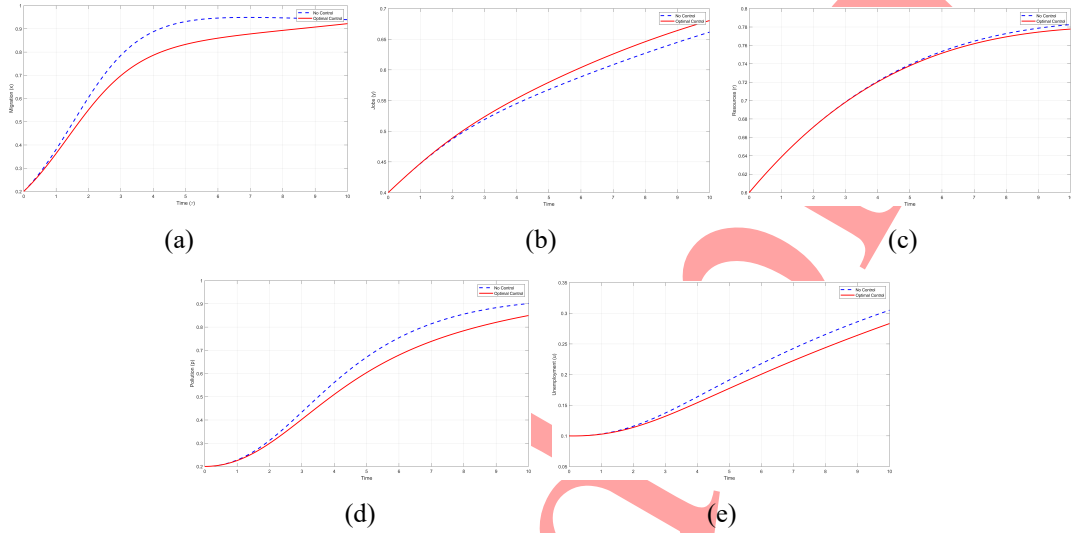


Figure 9: Time evolution of the state variables under Strategy A ($v_1 \neq 0$, $v_2 = 0$), showing increased employment, reduced unemployment and migration, and improved environmental conditions. The panels (a)–(e) show, respectively, migration $x(\tau)$, job availability $y(\tau)$, public resource availability $r(\tau)$, pollution $p(\tau)$ and unemployment $u(\tau)$.

Figure 10 depicts Strategy B, which imposes only environmental regulation. In this case, pollution levels drop substantially, but the underlying socio-economic pressures remain largely unaddressed.

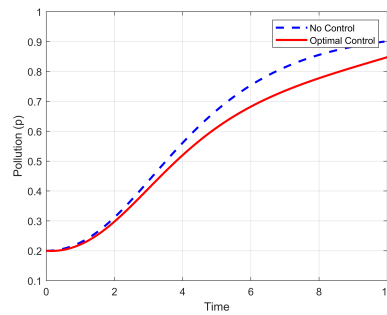


Figure 10: Pollution dynamics under Strategy B ($v_1 = 0$, $v_2 \neq 0$), showing effective suppression of pollution growth compared with the uncontrolled case.

Figure 11 demonstrates that the combined control strategy (Strategy C) leads to the best balanced outcome. The simultaneous application of both measures decreases migration, unemployment, and pollution while improving employment conditions. The resource level is slightly

lower under control, which suggests greater use of public infrastructure and shared resources associated with increasing employment and economic activity rather than a deterioration in the general economic state.

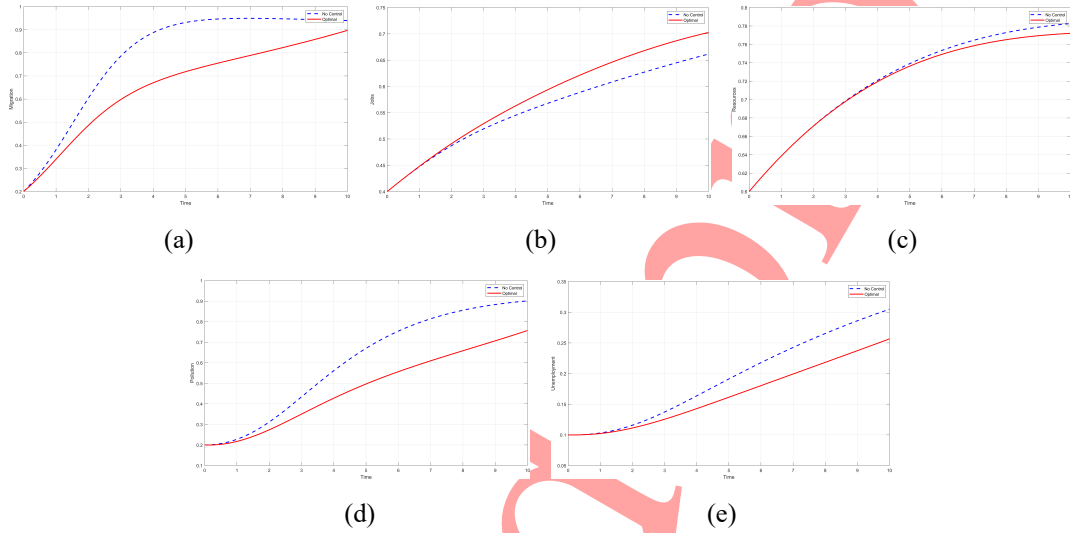


Figure 11: Time evolution under Strategy C ($v_1 \neq 0$, $v_2 \neq 0$), showing increased employment, reduced migration and unemployment, lower pollution, and changes in public resource utilization. The panels (a)–(e) show, respectively, migration $x(\tau)$, job availability $y(\tau)$, public resource availability $r(\tau)$, pollution $p(\tau)$ and unemployment $u(\tau)$.

The optimal control profiles in Figure 12 imply that migration regulation should begin at a moderate level, intensify in the intermediate stages to limit the migratory surge, and then diminish progressively over time. Pollution mitigation, in contrast, is applied more steadily in the early and middle periods before gradually decreasing toward the end of the control period.

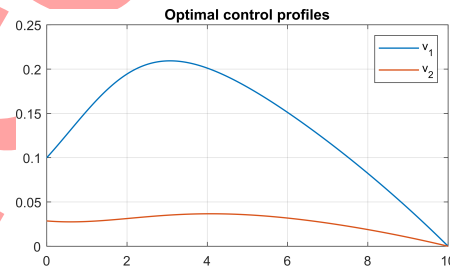


Figure 12: Optimal control profiles $v_1(\tau)$ and $v_2(\tau)$ obtained from the Pontryagin optimality system.

Overall, the numerical simulations corroborate the analytical findings by explicitly identifying four feasible endemic equilibria, establishing that only E_1^* is locally asymptotically stable for the selected parameter set, and demonstrating convergence toward this equilibrium within its corresponding stability domain. The simulations further illustrate the effectiveness of the

proposed control strategies in reducing migration pressure and environmental pollution while improving employment conditions.

9 Conclusion and Future Scope

In this work, a nonlinear migration–environment model was developed to investigate the coupled dynamics of migration pressure, unemployment, employment opportunities, public resources, and pollution within a unified framework. The mathematical analysis established the well-posedness of the system (Lemma 1) and confirmed the existence of feasible endemic equilibria representing long-term socio-economic and environmental states (Theorem 1). For the selected parameter set, four feasible endemic equilibria were identified, of which only one was found to be locally asymptotically stable (Theorem 2). The regional asymptotic stability analysis further established convergence toward the stable endemic equilibrium within the corresponding stability domain (Theorem 3).

The numerical simulations revealed the strong interdependence among migration, socio-economic conditions, and environmental quality. Increasing migration pressure exacerbates unemployment and pollution while also straining public resources. These interconnected dynamics emphasize the necessity of balanced policy intervention.

To regulate these dynamics, an optimal control framework was incorporated, using migration regulation and pollution mitigation as intervention strategies (Section 7). The analytical characterization of the controls (Theorem 5), combined with the numerical simulations (Section 8), revealed that the most effective outcome is obtained by the coordinated implementation of both controls, which reduces migration pressure and pollution while improving employment conditions and promoting a more balanced use of public resources. The findings indicate that sustainable regional management cannot rely on isolated measures but requires integrated socio-economic and environmental planning.

Overall, the study demonstrates how mathematical modeling can contribute to a better understanding of the delicate balance among human migration, economic stability, and environmental sustainability. These findings underline the relevance of a unified dynamical framework for comprehending the long-term interplay of human mobility, economic conditions, and environmental pressure.

The logistic representation of migration growth is a simplifying assumption that captures constrained migration pressure and may be extended with empirically validated migration functions. Furthermore, the model employs synthetic parameter values, so the resulting policy recommendations are qualitative rather than empirically calibrated; future data-driven parameter estimation could improve its predictive usefulness. The present model treats environmental pol-

lution as a consequence of migration and does not explicitly incorporate pollution as a direct determinant of migration decisions. Future studies might extend the model to include pollution-dependent migratory behavior. Time delays can be used to account for administrative response lags and delayed migration decisions, while stochastic effects can be used to represent economic uncertainty and environmental variability. Spatial diffusion and network-based structures may further describe migration among multiple interconnected regions. Finally, fractional-order formulations may improve the realism of the model for future policy assessments and sustainable development planning.

Appendix

A Derivation of the Cubic Equation for the Endemic Equilibrium

At the endemic equilibrium, the third equation of system (3) reads

$$ey^*(1 - y^*) - fx^* = 0,$$

which is equivalent to

$$ey^* - e(y^*)^2 - fx^* = 0.$$

To shorten the notation, set

$$\xi = \frac{d}{a}, \quad \zeta = b - \frac{d}{a}.$$

Then Equation (6) becomes

$$y^* = \frac{1}{c} [bx^* - \xi x^*(1 - x^*)] = \frac{1}{c} [(b - \xi)x^* + \xi x^{*2}] = \frac{x^*}{c} (\zeta + \xi x^*),$$

and hence

$$(y^*)^2 = \frac{x^{*2}}{c^2} (\zeta + \xi x^*)^2 = \frac{x^{*2}}{c^2} (\zeta^2 + 2\zeta\xi x^* + \xi^2 x^{*2}).$$

Substituting these expressions for y^* and $(y^*)^2$ into the equilibrium condition gives

$$\frac{ex^*}{c} (\zeta + \xi x^*) - \frac{ex^{*2}}{c^2} (\zeta^2 + 2\zeta\xi x^* + \xi^2 x^{*2}) - fx^* = 0.$$

Since $x^* > 0$ at the endemic equilibrium, dividing throughout by x^* yields

$$\frac{e}{c} (\zeta + \xi x^*) - \frac{ex^*}{c^2} (\zeta^2 + 2\zeta\xi x^* + \xi^2 x^{*2}) - f = 0.$$

Collecting the coefficients of like powers of x^* , we obtain

$$-\frac{e}{c^2} \xi^2 x^{*3} - \frac{2e}{c^2} \zeta\xi x^{*2} + \left[\frac{e}{c} \xi - \frac{e}{c^2} \zeta^2 \right] x^* + \frac{e}{c} \zeta - f = 0.$$

Returning to the original parameters, this is the cubic equation

$$A_3 x^{*3} + A_2 x^{*2} + A_1 x^* + A_0 = 0,$$

with

$$\begin{aligned} A_3 &= -\frac{e}{c^2} \left(\frac{d}{a} \right)^2, \\ A_2 &= -\frac{2e}{c^2} \left(b - \frac{d}{a} \right) \frac{d}{a}, \\ A_1 &= \frac{e}{c} \frac{d}{a} - \frac{e}{c^2} \left(b - \frac{d}{a} \right)^2, \\ A_0 &= \frac{e}{c} \left(b - \frac{d}{a} \right) - f. \end{aligned}$$

Hence, the coefficients A_0 – A_3 in Equation (7) follow directly by substituting the equilibrium expression for y^* into the third equilibrium equation and collecting the corresponding powers of x^* .

B Derivation of the Characteristic Polynomial of the Socio-Economic Subsystem

The Jacobian matrix corresponding to the socio-economic subsystem (x, u, y) at the endemic equilibrium is

$$\mathbf{J}_{3 \times 3} = \begin{pmatrix} 1 - 2x^* & -a & 0 \\ b & -d & -c \\ -f & 0 & e(1 - 2y^*) \end{pmatrix}.$$

To shorten the notation, set

$$\omega_1 = 1 - 2x^*, \quad \omega_2 = e(1 - 2y^*).$$

The characteristic equation is obtained from $\det(\lambda \mathbf{I} - \mathbf{J}_{3 \times 3}) = 0$, that is,

$$\begin{vmatrix} \lambda - \omega_1 & a & 0 \\ -b & \lambda + d & c \\ f & 0 & \lambda - \omega_2 \end{vmatrix} = 0.$$

Expanding the determinant along the first row gives

$$\det(\lambda \mathbf{I} - \mathbf{J}_{3 \times 3}) = (\lambda - \omega_1)(\lambda + d)(\lambda - \omega_2) + ab(\lambda - \omega_2) + acf.$$

Now,

$$(\lambda + d)(\lambda - \omega_2) = \lambda^2 + (d - \omega_2)\lambda - d\omega_2,$$

and therefore

$$\begin{aligned} (\lambda - \omega_1)(\lambda + d)(\lambda - \omega_2) &= \lambda^3 + (d - \omega_1 - \omega_2)\lambda^2 \\ &\quad + (\omega_1\omega_2 - d\omega_1 - d\omega_2)\lambda + d\omega_1\omega_2. \end{aligned}$$

Adding the remaining terms $ab(\lambda - \omega_2) + acf$ and collecting like powers of λ , we obtain

$$\begin{aligned} \det(\lambda \mathbf{I} - \mathbf{J}_{3 \times 3}) &= \lambda^3 + (d - \omega_1 - \omega_2)\lambda^2 \\ &\quad + (\omega_1\omega_2 - d\omega_1 - d\omega_2 + ab)\lambda \\ &\quad + d\omega_1\omega_2 - ab\omega_2 + acf. \end{aligned}$$

Thus, $\det(\lambda \mathbf{I} - \mathbf{J}_{3 \times 3}) = \lambda^3 + R_1\lambda^2 + R_2\lambda + R_3$, where

$$\begin{aligned} R_1 &= d - \omega_1 - \omega_2 = d - (1 - 2x^*) - e(1 - 2y^*), \\ R_2 &= \omega_1\omega_2 - d\omega_1 - d\omega_2 + ab \\ &= -d(1 - 2x^*) - de(1 - 2y^*) + (1 - 2x^*)e(1 - 2y^*) + ab, \\ R_3 &= d\omega_1\omega_2 - ab\omega_2 + acf \\ &= (1 - 2x^*)de(1 - 2y^*) - abe(1 - 2y^*) + acf. \end{aligned}$$

Hence, the characteristic equation corresponding to the subsystem (x, u, y) is Equation (9), namely

$$\lambda^3 + R_1\lambda^2 + R_2\lambda + R_3 = 0.$$

Declarations

Availability of Supporting Data

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Conflict of Interest

The authors declare that they have no conflict of interest.

Authors' Contributions

Champak Nath: Conceptualization; Methodology; Formal analysis; Writing – original draft preparation. **Ranu Paul:** Conceptualization; Methodology; Writing – review & editing; Supervision.

Artificial Intelligence Statement

Both authors used generative AI (ChatGPT) solely for language refinement, grammar correction, and improving readability. No AI tool was used for mathematical derivations, scientific analysis, data generation, interpretation of results, or scientific conclusions.

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