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Topp-Leone Distribution: A Reliability Analysis Perspective

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Abstract. This article presents a comprehensive investigation into the structural and reliability properties of the Topp-Leone distribution. A primary contribution of this study is the derivation of an exact finite series representation for the probability density function of order statistics, which rectifies previous approximation-based approaches. Furthermore, closed-form expressions for key information-theoretic measures, including Shannon entropy, extropy, and divergence metrics (e.g., Kullback-Leibler and Rényi), are established to quantify uncertainty and distributional disparity. The applicability of the TL distribution in system reliability is also examined through the stress-strength model. Collectively, these analytical results elucidate the theoretical foundations of the TL distribution and underscore its potential in statistical inference and reliability engineering contexts.

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1 Introduction

In the realm of practical research, the **precise** analysis of statistical distributions is paramount. These distributions offer invaluable insights into the occurrence and patterns. Central to this analysis is the information measures. These quantitative metrics provide a precise and objective means of quantifying the amount of information encapsulated within a dataset, thereby facilitating a deeper understanding of the underlying phenomena.

Information measures are mathematical functions that quantify the uncertainty or randomness associated with a random variable. In the context of statistical distributions, these measures assess how much new information is gained upon observing a particular outcome. Some of the most commonly employed information measures include entropy, extropy, and divergence measures. In contrast, the stress-strength model and the distribution of order statistics are important statistical and reliability concepts that can be analyzed using such information-theoretic quantities. Notable studies in this area include works by [16] which provide comprehensive insights into the application and implications of these measures.

The Topp-Leone (TL) distribution was originally introduced by [24] as a model for failure data, characterized by a bounded support over the unit interval $(0, 1)$ and a J-shaped probability density function (PDF). For several decades, this distribution attracted little attention; however, its recent application to bounded lifetime data has renewed interest.

By treating the random variable as the probability of an event, rather than deriving a probability distribution, we explore the distribution of an event modeled by a TL distribution with parameter θ [24]. This distribution recently has been modified by researchers to improve its suitability for modeling data. For example see [1] and [2]. The TL distribution, characterized by a shape parameter $\theta > 0$, has its cumulative distribution function (CDF) and PDF defined as follows [2, 7] and [19]:

$$F(x;\theta) = x^{\theta}(2-x)^{\theta}, \quad 0 < x < 1, \tag{1}$$

and

$$f(x;\theta) = 2\theta \cdot x^{\theta-1}(1-x)(2-x)^{\theta-1}, \quad 0 < x < 1. \tag{2}$$

The foundational properties of the TL distribution, including its moments and characteristic functions, were rigorously revisited by [7]. Their work demonstrated the mathematical tractability of the distribution, thereby facilitating its subsequent applications.
In the context of reliability analysis, [8] provided a detailed investigation into the hazard rate functions and stochastic ordering of the TL distribution. They demonstrated that the TL distribution exhibits a bathtub-shaped hazard rate, making it a viable competitor to the Beta and Kumaraswamy distributions in reliability engineering.

Some notable extensions of the TL distribution in the literature include the new extended TL distribution developed by various researchers. These advancements build upon the original framework, incorporating additional parameters to enhance flexibility and applicability in different contexts. For further details, please refer to [24].
To overcome the limitation of the bounded support $(0, 1)$, significant efforts have been made to generalize the TL distribution. [21] introduced the TL generated family of distributions, allowing the TL distribution to serve as a generator for new distributions with unbounded support. This innovation led to the development of the TL-Weibull and TL-Normal distributions, which offer greater flexibility in modeling real-world data with heavy tails.

Similarly, [22] proposed the TL generalized exponential distribution, discussing its estimation methods and application to lifetime data. These generalizations address the limitation of the bounded support by extending the domain and applicability of the base model.
In the domain of information theory and divergence criteria, [5] conducted a study on the entropy of the TL distribution. They derived expressions for Shannon entropy and Rényi entropy, providing tools to measure the uncertainty associated with the random variable.

Finaly, numerous relevant studies on the TL distribution are studied. For example, [8] explored reliability aspects. [9] investigated the kurtosis of the TL model. [13] examined the order statistics of this distribution. Interval estimation for the TL distribution was discussed by [20].

While the TL distribution has received considerable attention in the statistical literature, previous studies have mainly emphasized its basic structural properties and the moments of order statistics. In particular, [15] discussed moment-related properties of J-shaped distributions, while [5] and [13] provided dedicated investigations of the moments of order statistics for the TL distribution. These contributions constitute an important foundation for subsequent work. However, a comparatively limited effort has been made to present these distributional and order-statistical properties alongside a systematic treatment of information-theoretic quantities and analytical reliability measures within one unified framework. Motivated by this gap, the present paper brings together order statistics, entropy/extropy measures, divergence-based criteria, and stress–strength reliability for the TL model. Thus, the contribution of this study lies not in reintroducing the general theory of order statistics, but in integrating these complementary analytical components into a coherent treatment that broadens the interpretive and applied scope of the distribution. The remainder of this paper is organized as follows. In Section 2, we derive key mathematical properties, including probability-weighted moments (PWMs), quantile function approach, and order statistics distributions. We compare PWM estimators with maximum likelihood estimators (MLE) using bias and mean squared error (MSE) criteria. Additionally, we propose an estimator for the PDF of order statistics via binomial series expansion. This approximation is then compared with the exact PDF of the rth order statistic. Furthermore, we evaluate the binomial series-based estimator against the quantile function approach by examining bias and MSE relative to the TL PDF and CDF.

Section 3 presents analytic derivations for Shannon entropy, extropy, Kullback-Leibler (KL) divergence, Rényi divergence, Bhattacharyya divergence and stress-strength reliability. A primary objective of our divergence measure analysis is to quantify the discrepancy between two TL distributions with distinct parameters. This investigation provides novel insights into the sensitivity of divergence measures within a parametric family, particularly in assessing how distributional variations affect divergence. We also explicitly examine this framework in the context of stress-strength probability, $P(X < Y)$, where X and Y follow TL distributions with differing parameters.

Concluding remarks and future research directions are discussed in Section 4.

2 Some Properties of TL Distribution

In this section, we investigate the fundamental structural properties of the TL distribution, including the behavior of its PDF and CDF. We obtain closed-form expressions for the nth moments and PWMs. Using these moments, we obtain closed-form formulations for the mean

and variance. Furthermore, we derive the quantile function and the exact distribution of the order statistics.

2.1 The Characteristics of PDF and CDF of TL Distribution

The shape of the TL distribution is strictly governed by the parameter θ. Based on the value of θ, the behavior of the PDF, $f(x)$, can be categorized into three distinct regimes:
1. Case $\theta = 1$: The distribution reduces to the Beta$(1, 2)$ distribution. In this case, the PDF simplifies to a linear decreasing function $f(x) = 2(1 - x)$, taking the value of 2 at $x = 0$ and vanishing at $x = 1$.
2. Case $0 < \theta < 1$: The PDF exhibits a reversed J-shape. It has a singularity at the origin, such that $\lim_{x\to 0^+} f(x) = \infty$, and decreases monotonically to zero as $x \to 1$. This implies a high probability of observing small values (early failures in a reliability context).
3. Case $\theta > 1$: The distribution becomes unimodal with $f(0) = 0$ and $f(1) = 0$. As θ increases, the mode of the distribution shifts towards the right (i.e., towards $x = 1$), and the probability mass becomes increasingly concentrated, resulting in negative skewness.
Regarding the CDF, the parameter θ dictates the rate of probability accumulation. Specifically, for $\theta < 1$, the CDF rises steeply near $x = 0$, indicating rapid initial accumulation. Conversely, for large θ ($\theta > 1$), the CDF remains flat for small x and increases sharply only as x approaches 1. This flexibility allows the TL distribution to model diverse failure rates effectively.
Figures 1 and 2 show a graphical representation of PDF and CDF respectively for the TL Distribution with varying θ values.

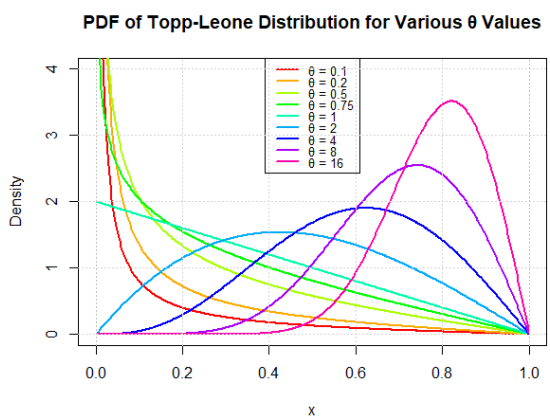

Figure 1: Graphical representation of PDF for the TL Distribution with varying θ values.

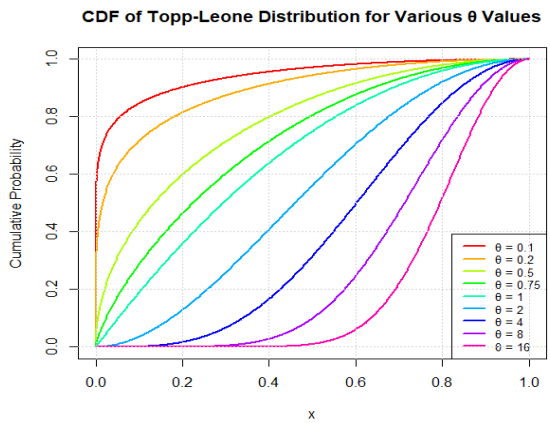

Figure 2: Graphical representation of CDF for the TL Distribution with varying θ values.

2.2 Probability-Weighted Moments

Probability-weighted moments (PWMs) are statistical measures that involve the expectations of specific functions of a random variable. They can be calculated for any random variable with finite moments. A particularly useful application of PWMs is in parameter estimation for

distributions whose inverse CDF lacks a closed-form expression.

For a random variable X following a TL distribution, the $(s,r)^{th}$ PWM, denoted as $\rho_{s,r}$ is defined as:

$$\rho_{s,r} = E[X^s F(x)^r] = \int_0^1 f(x) x^s F(x)^r dx, \tag{3}$$

where $E[\cdot]$ denotes the expectation operator. Then

$$\rho_{s,r} = \int_0^1 2\theta x^{\theta(r+1)+s-1}(1-x)(2-x)^{\theta(r+1)-1} dx = \frac{1}{r+1} E_{\theta(r+1)}(X^s), \tag{4}$$

where $E_{\theta(r+1)}(X)$ denotes the expectation value of the TL distribution with the parameter $\theta(r+1)$.

Example 1. Let $X \sim \mathrm{TL}(1)$, where $\mathrm{TL}(\theta)$ denotes the TL distribution with parameter $\theta = 1$. The PDF and CDF are given by $f(x) = 2(1-x)$ and $F(x) = x(2-x)$, respectively. The function $\rho_{(s,1)}(X)$ is defined as:

$$\rho_{(s,1)}(X) = \int_0^1 2x^{s+1}(1-x)(2-x)\, dx.$$

Simplifying the integrand gives:

$$\rho_{(s,1)}(X) = 4\int_0^1 x^{s+1}(1-x)\, dx - 2\int_0^1 x^{s+2}(1-x)\, dx.$$

Recognizing the integrals as Beta functions $B(a,b)$, specifically $\int_0^1 x^k(1-x)dx = \frac{1}{(k+1)(k+2)}$:

$$\rho_{(s,1)}(X) = \frac{4}{(s+2)(s+3)} - \frac{2}{(s+3)(s+4)}.$$

After algebraic simplification, the closed form is:

$$\rho_{(s,1)}(X) = \frac{2(s+6)}{(s+2)(s+3)(s+4)}.$$

Example 2. Let $X \sim \mathrm{TL}(\theta)$. The function $\rho_{(1,r)}(X)$ denoted by β_r, as:

$$\beta_r = E[X\{F(X)\}^r] = \int_0^1 x[F(x)]^r f(x)\, dx.$$

Using the substitution $u = F(x)$, we can express this integral in terms of the quantile function $x(u) = F^{-1}(u)$.

$$x(u) = 1 - \sqrt{1-u^{1/\theta}}, \quad 0 < u < 1.$$

Substituting this into the definition of β_r, we obtain:

$$\beta_r = \int_0^1 \left(1 - \sqrt{1-u^{1/\theta}}\right) u^r\, du.$$

This integral can be separated into two components:

$$\beta_r = \int_0^1 u^r\, du - \int_0^1 u^r (1-u^{1/\theta})^{1/2}\, du.$$

The first term trivially integrates to $\frac{1}{r+1}$. For the second term, we employ the substitution $z = u^{1/\theta}$, which implies $u = z^\theta$ and $du = \theta z^{\theta-1} dz$. The limits of integration remain 0 to 1. The integral transforms as follows:

$$\int_0^1 z^{\theta r}(1-z)^{1/2} \cdot \theta z^{\theta-1}\, dz = \theta \int_0^1 z^{\theta(r+1)-1}(1-z)^{1/2}\, dz.$$

We recognize this integral as the standard Euler Beta function, $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$, where $a = \theta(r+1)$ and $b = 3/2$.

Thus, the explicit form for the PWMs of the TL distribution is:

$$\beta_r = \frac{1}{r+1} - \theta B\left(\theta(r+1), \frac{3}{2}\right).$$

where $B(\cdot,\cdot)$ represents the Beta function. This closed-form expression facilitates the estimation of the parameter θ via the method of moments.

2.3 Parameter Estimation and Simulation Study

In this subsection, we derive the estimators for the parameter θ using the Maximum Likelihood (ML) method and the Probability-Weighted Moments (PWM) method. Subsequently, we perform a Monte Carlo simulation to compare their performance in terms of Bias and Mean Squared Error (MSE).

2.3.1 Maximum Likelihood Estimator (MLE)

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the TL distribution. The likelihood function is given by:

$$L(\theta) = \prod_{i=1}^{n} 2\theta(1-x_i)[x_i(2-x_i)]^{\theta-1}.$$

The log-likelihood function, $\ell(\theta) = \ln L(\theta)$, is:

$$\ell(\theta) = n\ln(2) + n\ln(\theta) + \sum_{i=1}^{n} \ln(1-x_i) + (\theta-1)\sum_{i=1}^{n} \ln(x_i(2-x_i)).$$

By differentiating $\ell(\theta)$ with respect to θ and setting the result to zero, we obtain the closed-form estimator: $\frac{\partial \ell(\theta)}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \ln(x_i(2-x_i)) = 0$ implies

$$\hat{\theta}_{MLE} = \frac{-n}{\sum_{i=1}^{n} \ln(x_i(2-x_i))}.$$

2.3.2 Probability-Weighted Moments Estimator (PWM)

The PWM method equates the sample statistic to their population counterparts. The first PWM (β_0) and the second PWM (β_1) are used to isolate θ. The estimator $\hat{\theta}_{PWM}$ is obtained by solving

the system of equations derived from Eq. (5) numerically or by using the analytical relation between the first moment and θ. While the ML estimator possesses a closed-form solution in this specific one-parameter case, the PWM method is often preferred in small samples due to its robustness against outliers.

2.3.3 Simulation Results

To evaluate the finite-sample performance of the estimators, we conducted a **simulation study**. Samples of sizes $n = 5, 10, 20, 30, 50, 100, 200$ were generated from the TL distribution with $\theta \in \{0.5, 1.0, 2.0\}$. The simulation was replicated $N = 1,000$ times. The empirical Bias and MSE were calculated as:

$$\text{Bias} = \frac{1}{N}\sum_{j=1}^{N}(\hat{\theta}_j - \theta), \quad \text{MSE} = \frac{1}{N}\sum_{j=1}^{N}(\hat{\theta}_j - \theta)^2.$$

The simulation results, illustrated in Figure 3, reveal an explicit bias-variance trade-off between the two estimation methods. As shown in the figure, the PWM estimator (red line) consistently demonstrates lower bias compared to the MLE (blue line). This indicates that for limited data, the PWM method provides estimates that are, on average, closer to the true parameter value, thereby offering higher accuracy in terms of mean deviation. It is worth noting that the bias for both methods converges to zero as n increases, empirically confirming that both estimators are asymptotically unbiased and consistent.

Conversely, the results demonstrate that the MLE generally achieves a lower Mean Squared Error (MSE) than the PWM estimator across almost all scenarios. Recalling the decomposition $\text{MSE} = \text{Var}(\hat{\theta}) + [\text{Bias}(\hat{\theta})]^2$, this finding implies that although the MLE carries a slightly higher bias in small samples, it possesses significantly lower variance compared to PWM. In other words, the MLE offers higher precision, which compensates for its bias, aligning with the theoretical asymptotic efficiency of maximum likelihood estimation.

Consequently, the choice between these estimators depends on the practitioner's specific objective. If minimizing systematic error (bias) is the priority in small samples ($n \leq 20$), PWM is preferable. However, if the goal is to minimize total error variability, MLE remains the superior choice due to its higher efficiency. For moderate to large samples ($n > 50$), the difference becomes negligible, and both methods perform comparably well.

2.4 Order statistics

Order statistics are fundamental in many statistical applications. Consider a random sample $X_1, X_2, \ldots, X_n$ drawn from a TL distribution. Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ denote the corresponding order statistics arranged in ascending order. The PDF of the r-th order statistic, $X_{r:n}$, is given by:

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x), \quad 0 < x < 1. \tag{5}$$

Substituting the PDF and CDF of the TL distribution, we obtain:

$$f_{r:n}(x) = \frac{n!2\theta(1-x)\left[(1-(1-x)^2)^\theta\right]^{r-1}\left[1-(1-(1-x)^2)^\theta\right]^{n-r}\left[1-(1-x)^2\right]^{\theta-1}}{(r-1)!(n-r)!}. \tag{6}$$

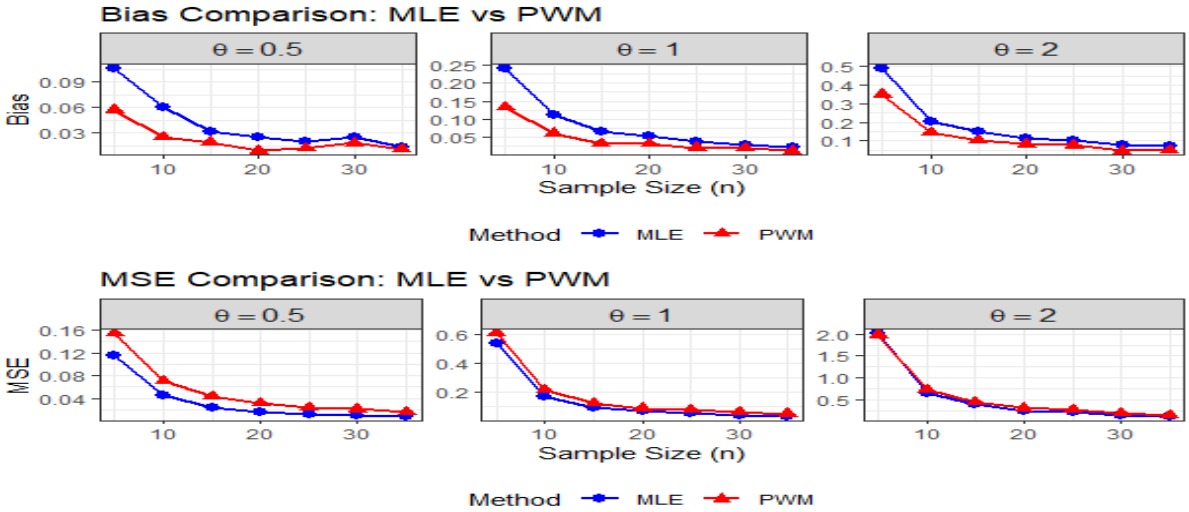

Figure 3: Comparing Bias and MSE for MLE and PWM across sample sizes, with separate lines for $\theta = 0.5, 1.0, 2.0$.

Proposition 1. If θ is a positive integer, the PDF of the r-th order statistic, $f_{r:n}(x)$, can be expressed exactly as a finite mixture of polynomial terms:

$$f_{r:n}(x) = C_{r,n} \sum_{j=0}^{n-r} \sum_{k=0}^{\theta(r+j)-1} w_{j,k}(1-x)^{2k+1}, \tag{7}$$

where $C_{r,n} = \frac{n!2\theta}{(r-1)!(n-r)!}$ and the coefficients $w_{j,k}$ are defined as:

$$w_{j,k} = (-1)^{j+k} \binom{n-r}{j} \binom{\theta(r+j)-1}{k}. \tag{8}$$

Proof. Let $y = 1-(1-x)^2$. Then $F(x) = y^\theta$ and $f(x) = 2\theta(1-x)y^{\theta-1}$. The PDF of the r-th order statistic becomes:

$$\begin{aligned} f_{r:n}(x) &= C'_{r,n}[y^\theta]^{r-1}[1-y^\theta]^{n-r} \cdot 2\theta(1-x)y^{\theta-1} \\ &= C_{r,n}(1-x)y^{\theta(r-1)+\theta-1}[1-y^\theta]^{n-r} \\ &= C_{r,n}(1-x)y^{\theta r-1}[1-y^\theta]^{n-r}. \end{aligned}$$

Using the binomial expansion for $[1-y^\theta]^{n-r}$ (which is a finite sum since $n-r$ is an integer):

$$[1-y^\theta]^{n-r} = \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j y^{\theta j}.$$

Substituting this back:

$$\begin{aligned} f_{r:n}(x) &= C_{r,n}(1-x) \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j y^{\theta r-1+\theta j} \\ &= C_{r,n}(1-x) \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j \left[1-(1-x)^2\right]^{\theta(r+j)-1}. \end{aligned}$$

Since θ is an integer, the exponent $M_j = \theta(r+j) - 1$ is also an integer. We can apply the binomial expansion again to $\left[1-(1-x)^2\right]^{M_j}$:

$$\left[1-(1-x)^2\right]^{M_j} = \sum_{k=0}^{M_j} \binom{M_j}{k} (-1)^k \left((1-x)^2\right)^k = \sum_{k=0}^{M_j} \binom{M_j}{k} (-1)^k (1-x)^{2k}.$$

Finally, combining all terms:

$$f_{r:n}(x) = C_{r,n} \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j \sum_{k=0}^{\theta(r+j)-1} \binom{\theta(r+j)-1}{k} (-1)^k (1-x)^{2k+1}.$$

This completes the proof. □

Example 3. To rigorously validate the analytical derivation presented in Proposition 1, we performed a comparative numerical analysis using the exact finite series representation derived in Equation (7). The validation was conducted for a sample size of $n = 5$ and the second order statistic ($r = 2$) across a set of integer shape parameters $\theta \in \{1, 2, 4, 6\}$. The integer constraint is necessary as Proposition 1 relies on finite binomial expansions.
Figure 4 displays the PDFs computed via two distinct methods:

1. Standard Definition (Solid, Thick Lines): Computed using the numerical evaluation of $f_{r:n}(x)$ based on the standard PDF and CDF given in Eq. (22).
2. Proposed Exact Series (Dashed, Thin Lines): Computed using the explicit algebraic sum derived in Eq. (7), which utilizes the double summation of polynomial terms weighted by coefficients $w_{j,k}$.

As demonstrated in Figure 4, the proposed series approximation perfectly overlays the standard definition curves for all tested values of θ. The perfect agreement between the two methods confirms the correctness of the combinatorial coefficients $C_{r,n}$ and $w_{j,k}$, as well as the polynomial powers used in the derivation. This validates that for integer θ, the order statistic PDF can be expressed exactly as a finite sum of polynomial terms, facilitating simplified integration for subsequent moment calculations.

Remark 1. For an integer parameter θ, the exact PDF and CDF of the TL distribution admit finite series representations. By setting $u = 1 - x$ and applying the binomial expansion, derived as a special case of Proposition 1, we obtain:

$$f(x) = 2\theta \sum_{k=0}^{\theta-1} (-1)^k \binom{\theta-1}{k} u^{2k+1}, \tag{9}$$

$$F(x) = \sum_{k=0}^{\theta} (-1)^k \binom{\theta}{k} u^{2k}. \tag{10}$$

2.5 Quantile-Based Shape Analysis and Risk Measures

The quantile function, $Q(p)$, serves as a convenient alternative to standard moment-based methods for analyzing the distributional properties, particularly when classical moments involve

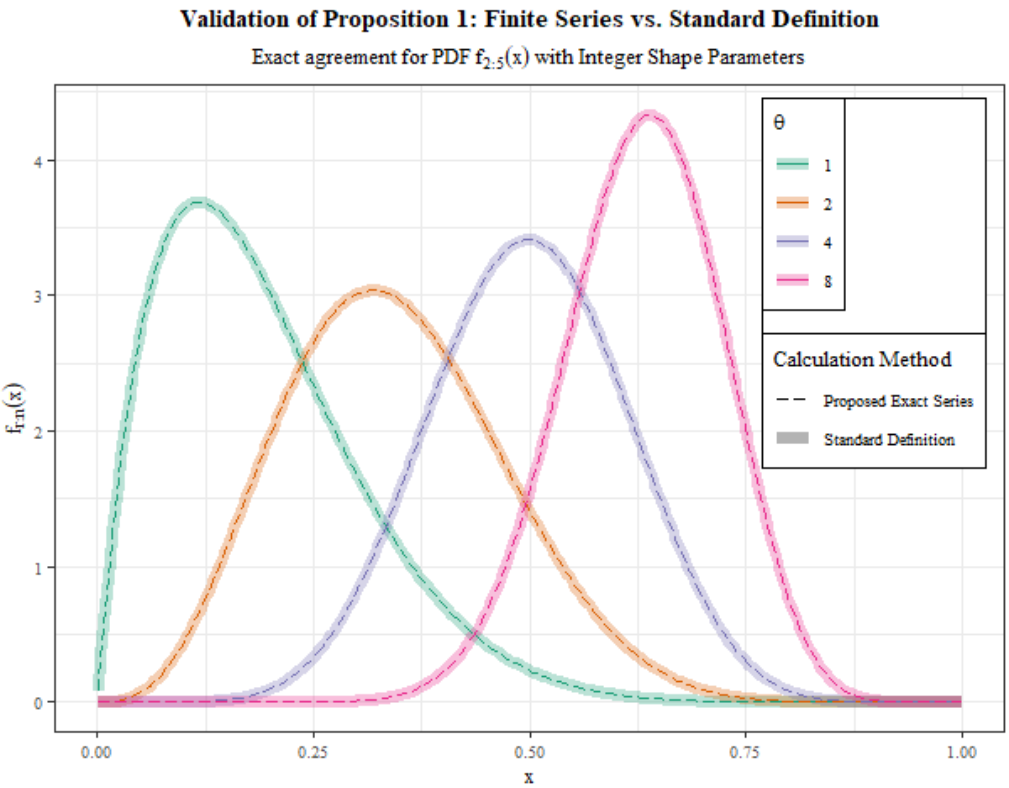

Figure 4: $n = 5, r = 2$. The dashed lines (representing the derived formula in Eq. 7) strictly coincide with the thick solid lines (representing the standard definition), confirming the exactness of the finite series representation for integer θ.

complex special functions. [8] **studied** some properties of TL quantile approach.
Inverting the CDF in Eq. (2), the quantile function of TL distribution is given by:

$$Q(p) = F^{-1}(p) = 1 - \sqrt{1 - p^{1/\theta}}, \quad 0 \leq p \leq 1. \tag{11}$$

The corresponding quantile density function is given by $q(p) = \frac{d}{dp}Q(p) = \frac{p^{(1/\theta)-1}}{2\theta\sqrt{1-p^{1/\theta}}}$.

2.5.1 Algebraic Measures of Skewness and Kurtosis

While the classical skewness and kurtosis of the TL distribution rely on higher-order moments involving complicated Gamma functions, quantile-based measures offer explicit algebraic representations that are computationally efficient and easier to interpret. We employ [4] skewness (S) and [14] kurtosis (K):

$$S = \frac{Q(3/4) - 2Q(2/4) + Q(1/4)}{Q(3/4) - Q(1/4)}, \tag{12}$$

$$K = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)}. \tag{13}$$

By substituting Eq. (11) into these expressions, one obtains simple closed-form metrics to monitor the transition of the TL density from a J-shaped to a bell-shaped curve as θ varies, without the need for numerical integration or evaluating high-order Gamma terms.

2.5.2 Application in Risk Management: TVaR

In reliability and financial contexts where variables are bounded (e.g., efficiency ratios or normalized losses), the *Value at Risk* ($VaR_p = Q(p)$) is a standard metric. However, to capture the expected loss strictly exceeding a threshold, we derive the *Tail Value at Risk* (TVaR). For the TL distribution, TVaR has a specific analytical form:

$$\text{TVaR}_p = \frac{1}{1-p} \int_p^1 \left(1 - \sqrt{1 - u^{1/\theta}}\right) du. \tag{14}$$

2.5.3 Numerical Illustration

To demonstrate the numerical efficiency of the quantile-based measures, we computed Galton's skewness (S) and Moors' kurtosis (K) for various values of the shape parameter θ. The results are summarized in Table 1.

As presented in Table 1, the quantile measures effectively capture the structural transition of the TL distribution. For $\theta \leq 1$, the distribution exhibits positive skewness ($S > 0$), consistent with its reversed J-shape or triangular decreasing nature, where the probability mass is concentrated near zero. However, as θ increases ($\theta > 1$), the skewness coefficient decreases and eventually becomes negative ($S < 0$). This transition quantitatively confirms the shift in the probability mass towards the upper bound (near $x = 1$), resulting in a left-skewed unimodal density.

Furthermore, the kurtosis value (K) increases monotonically with θ. Starting from approximately 1.64 for the bathtub shape, it rises as the distribution transforms into a peaked unimodal form. This monotonic increase in K indicates that for larger values of θ, the distribution becomes more peaked with heavier tails relative to its center, confirming the flexibility of the proposed model in fitting data with varying degrees of tail weight.

Table 1: Galton Skewness (S) and Moors Kurtosis (K) for the TL distribution with varying θ.

Parameter (θ)	**Shape Description**	**Skewness** (S)	**Kurtosis** (K)
0.5	Bathtub / U-shaped	0.2396	1.6378
1.0	Triangular (Decreasing)	0.1317	1.9519
2.0	Unimodal (Right-Skewed)	0.0381	2.2783
5.0	Unimodal (Left-Skewed)	-0.0573	2.5833
10.0	Unimodal (Left-Skewed)	-0.0963	2.7093

3 Information Measures

Shannon entropy, extropy, and divergence measures such as Kullback-Leibler (KL) divergence and Rényi divergence are fundamental concepts in information theory. These measures quantify the uncertainty, randomness, and dissimilarity between probability distributions. In this section, we will delve into the properties and bounds of these measures. Subsequently, we will explore the estimation of the TL distribution using the stress-strength model.

3.1 Entropy and Extropy

Entropy and extropy are fundamental concepts in information theory and statistics that describe uncertainty and information distribution within a system. Entropy, originally introduced by [23], quantifies the level of uncertainty or disorder in a probability distribution, representing the average amount of information obtainable from a random variable. In contrast, extropy, proposed by [11] as the complementary dual of entropy, measures the order or useful information, helping to illustrate how information is distributed among variables and its impact on decision-making. **[12] applied** these measures to generalized distributions, including the TL family, to assess reliability and uncertainty.

3.1.1 Shannon Entropy

Shannon entropy is a fundamental concept in information theory that measures the uncertainty or randomness inherent in a probability distribution. When applied to lifetime variables, it provides valuable insights into the unpredictability of component or system failure. Specially, the Shannon entropy for the TL distribution was obtained by [15]

Proposition 2. The Shannon entropy for the TL(θ) distribution is given by:

$$H(X) = 1 + \ln\left(\frac{1}{2\theta}\right) + \frac{\gamma + \psi(\theta) - \theta^{-1}}{2}, \tag{15}$$

where $\psi(\cdot)$ is the digamma function and $\gamma \approx 0.57721$ is the Euler–Mascheroni constant.

Proof. The Shannon entropy is defined as $H(X) = -E[\ln f(X)]$. The PDF of the TL distribution is $f(x) = 2\theta(1-x)[x(2-x)]^{\theta-1}$. The logarithm of the PDF is:

$$\ln f(x) = \ln(2\theta) + \ln(1-x) + (\theta-1)\ln\big(x(2-x)\big). \tag{16}$$

Taking the negative expectation of (16), we get:

$$H(X) = -\ln(2\theta) - E[\ln(1-X)] - (\theta-1)E\big[\ln(X(2-X))\big]. \tag{17}$$

Consider the transformation $T = X(2-X)$. Since $X \sim \mathrm{TL}(\theta)$, the random variable T follows a Beta distribution with parameters $\alpha = \theta$ and $\beta = 1$, denoted as $T \sim \mathrm{Beta}(\theta, 1)$.

Using the properties of the Beta distribution, we know that $E[\ln T] = \psi(\theta) - \psi(\theta+1)$. Using the recurrence relation $\psi(z+1) = \psi(z) + 1/z$, this simplifies to:

$$E\big[\ln(X(2-X))\big] = E[\ln T] = -\frac{1}{\theta}. \tag{18}$$

For the term $E[\ln(1-X)]$, note that $T = 2X - X^2 = 1 - (1-X)^2$, which implies $1 - X = (1-T)^{1/2}$. Therefore, $\ln(1-X) = \frac{1}{2}\ln(1-T)$. Since $T \sim \mathrm{Beta}(\theta, 1)$, the variable $1-T$ follows a $\mathrm{Beta}(1, \theta)$ distribution. Thus:

$$E[\ln(1-X)] = \frac{1}{2}E[\ln(1-T)] = \frac{1}{2}\big(\psi(1) - \psi(1+\theta)\big). \tag{19}$$

Knowing that $\psi(1) = -\gamma$ and $\psi(1+\theta) = \psi(\theta) + \frac{1}{\theta}$, Equation (19) becomes:

$$E[\ln(1-X)] = \frac{1}{2}\left(-\gamma - \psi(\theta) - \frac{1}{\theta}\right). \tag{20}$$

Substituting (18) and (20) back into (17):

$$\begin{aligned} H(X) &= -\ln(2\theta) - \frac{1}{2}\left(-\gamma - \psi(\theta) - \frac{1}{\theta}\right) - (\theta-1)\left(-\frac{1}{\theta}\right) \\ &= -\ln(2\theta) + \frac{\gamma + \psi(\theta)}{2} + \frac{1}{2\theta} + 1 - \frac{1}{\theta} \\ &= 1 + \ln\left(\frac{1}{2\theta}\right) + \frac{\gamma + \psi(\theta) - \theta^{-1}}{2}. \end{aligned} \tag{21}$$

This completes the proof. □

Example 4. To validate the exact closed-form expression derived for Shannon entropy, we performed a numerical analysis. The entropy values were computed for various shape parameters θ using two distinct methods: Shannon formula and Equation (15). Figure 5 illustrates the behavior of the Shannon entropy for $\theta \in [0.5, 10]$. The solid blue line represents the values obtained from the exact formula, while the red dots represent the results from numerical integration. The perfect overlap between the continuous curve and the discrete points corroborates the accuracy of our mathematical derivation.

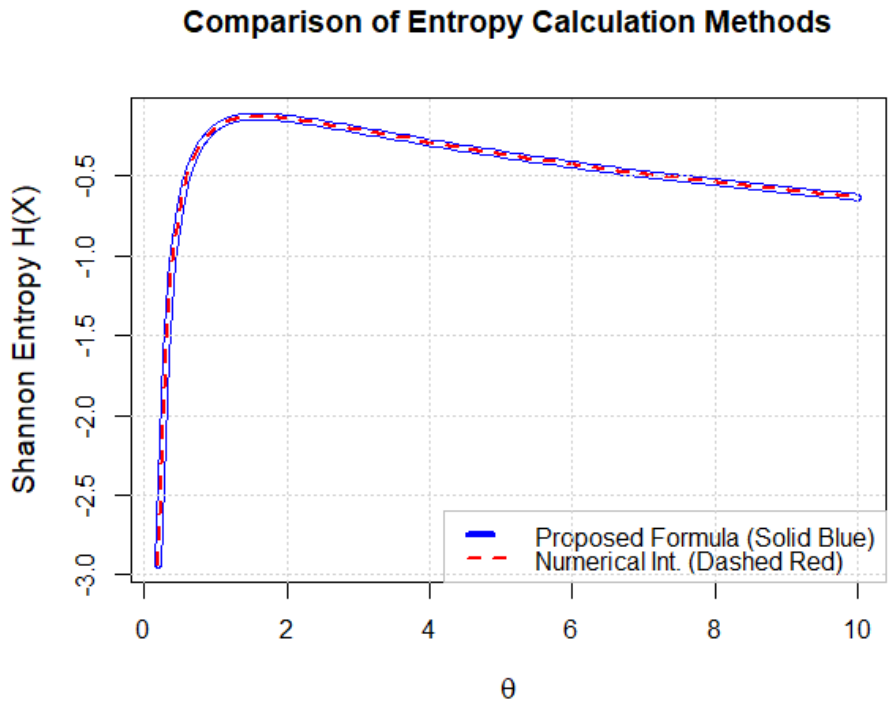

Figure 5: Comparison of the exact analytical Shannon entropy (solid blue line) versus numerical integration results (red dots) for the TL distribution.

3.1.2 Extropy Measure

Proposition 3. Let X be a random variable following the TL distribution with parameter $\theta > 0.5$. The extropy of X is derived based on the properties of the Beta function as follows:

$$J(X) = -\theta^2 B(2\theta - 1, 1.5), \tag{22}$$

where $B(\cdot, \cdot)$ denotes the Beta function. Note that for $\theta \leq 0.5$, the integral defining the extropy diverges, so the extropy is not finite.

Proof. Using the definition of extropy given by [3], and substituting the PDF of the TL distribution, we have:

$$J(X) = -2\theta^2 \int_0^1 (1-x)^2 \left(x(2-x)\right)^{2\theta-2} dx.$$

We employ the change of variable $u = x(2-x)$. Consequently, $du = 2(1-x)dx$. Also, noting that $(1-x)^2 = 1 - (x(2-x)) = 1 - u$, we can rewrite the term $(1-x)^2 dx$ as:

$$(1-x)^2 dx = (1-x) \cdot [(1-x)dx] = \sqrt{1-u} \cdot \frac{1}{2} du.$$

The limits of integration remain from 0 to 1. Substituting these into the integral:

$$\begin{aligned} J(X) &= -2\theta^2 \int_0^1 u^{2\theta-2} \cdot \frac{1}{2}(1-u)^{1/2} du \\ &= -\theta^2 \int_0^1 u^{(2\theta-1)-1}(1-u)^{1.5-1} du. \end{aligned}$$

Identifying the integral as the Beta function $B(\alpha, \beta)$ with $\alpha = 2\theta - 1$ and $\beta = 1.5$, we obtain the final result in Equation (22) and the proof is completed. □

Example 5. To verify the exactness of the closed-form expression derived in Equation (22), a numerical comparison was conducted between the integral definition of Lad extropy and the proposed formula based on the Beta function. The calculations were performed for the shape parameter θ varying in the interval $[1, 10]$.
Figure 6 illustrates the values obtained from direct numerical integration (solid black line) superimposed with the values calculated using the proposed formula (dashed red line). As evident from the figure, there is a perfect overlap between the two curves. The maximum absolute difference between the two methods was found to be less than 1×10^{-15}, which is attributed solely to machine floating-point precision. This result not only validates the accuracy of Proposition 3 but also demonstrates that the proposed closed-form expression is computationally more efficient and stable than numerical integration methods.

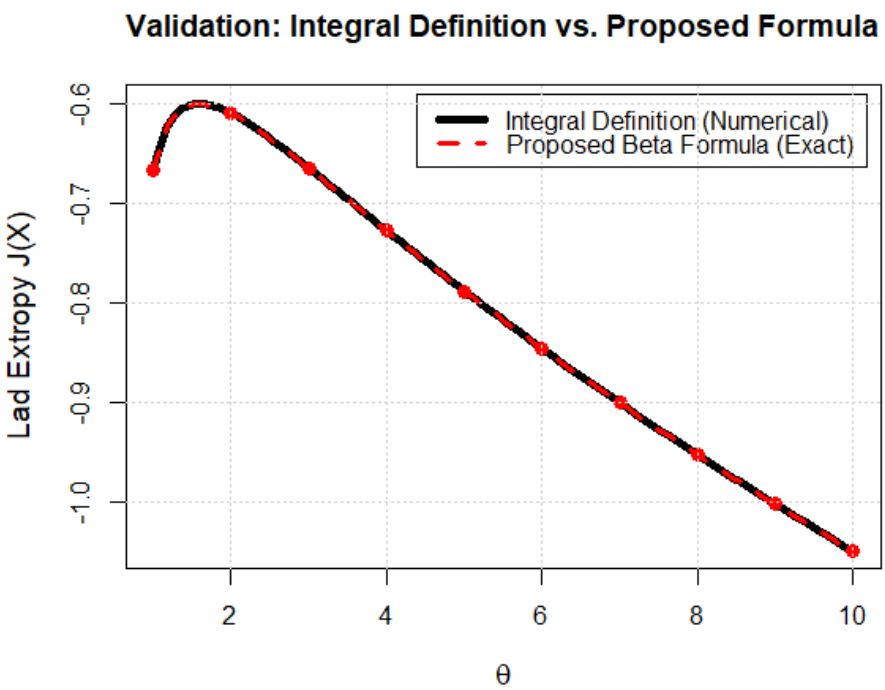

Figure 6: Comparison of Lad extropy values calculated using the integral definition (solid black line) and the proposed closed-form formula (dashed red line) for $\theta \in [1, 10]$. The perfect match validates the analytic derivation.

3.2 Divergence Measures

The impact of parameter changes on a distribution is a significant topic in statistics, particularly regarding the flexibility and shape of the distribution. Divergence measures serve as metrics to quantify how parameters influence distribution changes. Specifically, we can compute the divergence of a statistical distribution with respect to various parameter values, where a larger

divergence indicates a greater impact of the parameter on the distribution's flexibility and shape. In the following sections, we will analyze the effect of the parameter on the TL distribution, focusing specifically on the parameter θ. We will utilize the KL divergence and Rényi divergence as our measurement criteria, as these are well-known and widely used divergence measures.

3.2.1 Kullback-Leibler Divergence

KL divergence is a fundamental measure in information theory that quantifies the dissimilarity between two probability distributions. For continuous random variables, it provides a way to assess how one PDF diverges from a reference distribution [17]. Formally, given two continuous distributions P and Q with densities $p(x)$ and $q(x)$ respectively, the KL divergence from Q to P is defined as [10]:

$$D_{\mathrm{KL}}(P \parallel Q) = \int_{-\infty}^{\infty} p(x) \log\left(\frac{p(x)}{q(x)}\right) dx.$$

KL divergence can be a vital metric for evaluating the differences between two TL distributions defined by different parameters. It quantifies the information loss when one distribution approximates itself, when the parameter changed. Its applications span model evaluation, parameter estimation, and anomaly detection, providing insights into the behavior and shape of TL distributions. computational considerations and common pitfalls when applying it in practical settings.

Proposition 4. Let X_1 and X_2 be two continuous random variables following the TL distribution with parameters θ_1 and θ_2, respectively. If $\theta_1 \geq \theta_2$, an upper bound for the Kullback-Leibler (KL) divergence between the PDF's $f_{X_1}(x)$ and $f_{X_2}(x)$ is given by:

$$D_{\mathrm{KL}}(f_1||f_2) \leq \ln\left(\frac{\theta_1}{\theta_2}\right) + (\theta_1 - \theta_2) \ln\left(\frac{\theta_1}{\theta_1 + 1}\right). \tag{23}$$

Proof. By the definition of KL divergence, we have:

$$\begin{aligned} D_{\mathrm{KL}}(f_1||f_2) &= \int_0^1 f_{X_1}(x) \ln\left(\frac{f_{X_1}(x)}{f_{X_2}(x)}\right) dx \\ &= \int_0^1 f_{X_1}(x) \ln\left(\frac{2\theta_1(1-x)[x(2-x)]^{\theta_1-1}}{2\theta_2(1-x)[x(2-x)]^{\theta_2-1}}\right) dx \\ &= \int_0^1 f_{X_1}(x) \left[\ln\left(\frac{\theta_1}{\theta_2}\right) + (\theta_1 - \theta_2)\ln(x(2-x))\right] dx \\ &= \ln\left(\frac{\theta_1}{\theta_2}\right) + (\theta_1 - \theta_2) \underbrace{\int_0^1 f_{X_1}(x) \ln(x(2-x)) dx}_{E_{f_1}[\ln(x(2-x))]}. \end{aligned} \tag{24}$$

Using Jensen's inequality and assuming $\theta_1 \geq \theta_2$, we obtain:

$$\begin{aligned} D_{\mathrm{KL}}(f_1||f_2) &\leq \ln\left(\frac{\theta_1}{\theta_2}\right) + (\theta_1 - \theta_2) \ln\left(\int_0^1 f_{X_1}(x) \cdot (x(2-x)) dx\right) \\ &= \ln\left(\frac{\theta_1}{\theta_2}\right) + (\theta_1 - \theta_2) \ln\left(\int_0^1 2\theta_1(1-x)[x(2-x)]^{\theta_1-1} \cdot [x(2-x)] dx\right) \\ &= \ln\left(\frac{\theta_1}{\theta_2}\right) + (\theta_1 - \theta_2) \ln\left(\int_0^1 2\theta_1(1-x)[x(2-x)]^{\theta_1} dx\right). \end{aligned} \tag{25}$$

To solve the integral inside the logarithm, let $u = x(2-x)$, which implies $du = 2(1-x)dx$. The limits of integration remain from 0 to 1. Thus:

$$\int_0^1 \theta_1[x(2-x)]^{\theta_1}(2(1-x)dx) = \int_0^1 \theta_1 u^{\theta_1} du = \left[\frac{\theta_1}{\theta_1+1}u^{\theta_1+1}\right]_0^1 = \frac{\theta_1}{\theta_1+1}. \tag{26}$$

Substituting this result back into the inequality yields:

$$D_{\mathrm{KL}}(f_1||f_2) \leq \ln\left(\frac{\theta_1}{\theta_2}\right) + (\theta_1-\theta_2)\ln\left(\frac{\theta_1}{\theta_1+1}\right). \tag{27}$$

□

Proposition 5. Consider two continuous random variables, X_1 and X_2, following a TL distribution with parameters θ_1 and θ_2, respectively. Then, the KL divergence between the PDF's $f_{X_1}(x)$ and $f_{X_2}(x)$ approximately is given by:

$$D_{\mathrm{KL}}(f_1||f_2) = \ln\left(\frac{\theta_1}{\theta_2}\right) + \frac{(\theta_2-\theta_1)}{\theta_1}. \tag{28}$$

Proof. Substituting $x(2-x) = t$ and according to Equation (24) we have

$$\begin{aligned} D_{\mathrm{KL}}(f_1||f_2) &= \ln\left(\frac{\theta_1}{\theta_2}\right) + (\theta_1-\theta_2)\int_0^1 2\theta_1 \cdot x^{\theta_1-1}(1-x)(2-x)^{\theta_1-1}\ln(x(2-x))dx \\ &= \ln\left(\frac{\theta_1}{\theta_2}\right) + (\theta_1-\theta_2)\int_0^1 \theta_1 t^{\theta_1-1}\ln(t)dt \\ &= \ln\left(\frac{\theta_1}{\theta_2}\right) + (\theta_1-\theta_2)\big(\psi(\theta)-\psi(\theta+1)\big) \\ &= \ln\left(\frac{\theta_1}{\theta_2}\right) + \frac{(\theta_2-\theta_1)}{\theta_1}. \end{aligned} \tag{29}$$

□

3.2.2 Rényi Divergence

Proposition 6. Let X_1 and X_2 be two independent random variables following the TL distribution with parameters θ_1 and θ_2, respectively. For any $\alpha > 0$ and $\alpha \neq 1$, the Rényi divergence of order α between the PDFs $f_{X_1}(x)$ and $f_{X_2}(x)$ is given by:

$$D_\alpha(f_1\|f_2) = \frac{1}{\alpha-1}\ln\left(\frac{\theta_1^\alpha\theta_2^{1-\alpha}}{\alpha(\theta_1-\theta_2)+\theta_2}\right), \tag{30}$$

provided that $\alpha(\theta_1-\theta_2)+\theta_2 > 0$.

Proof. Using the definition of Rényi divergence, we have:

$$D_\alpha(f_1\|f_2) = \frac{1}{\alpha-1}\ln\left(\int_0^1 [f_{X_1}(x)]^\alpha\,[f_{X_2}(x)]^{1-\alpha}\,dx\right).$$

Substituting the PDF of the TL distribution:

$$\begin{aligned} I &= \int_0^1 \left[2\theta_1(1-x)(x(2-x))^{\theta_1-1}\right]^{\alpha} \left[2\theta_2(1-x)(x(2-x))^{\theta_2-1}\right]^{1-\alpha} dx \\ &= \int_0^1 2^{\alpha}\theta_1^{\alpha}(1-x)^{\alpha}[x(2-x)]^{\alpha(\theta_1-1)} \cdot 2^{1-\alpha}\theta_2^{1-\alpha}(1-x)^{1-\alpha}[x(2-x)]^{(1-\alpha)(\theta_2-1)}dx \\ &= 2\theta_1^{\alpha}\theta_2^{1-\alpha}\int_0^1 (1-x)[x(2-x)]^{\alpha(\theta_1-1)+(1-\alpha)(\theta_2-1)}dx. \end{aligned} \tag{31}$$

Let $K = \alpha(\theta_1-1)+(1-\alpha)(\theta_2-1)$. Simplifying the exponent, we get $K = \alpha\theta_1-\alpha+\theta_2-1-\alpha\theta_2+\alpha = \alpha(\theta_1-\theta_2)+\theta_2-1$.

To solve the integral $\int_0^1(1-x)[x(2-x)]^K dx$, we use the substitution $u = x(2-x)$, which implies $du = 2(1-x)dx$ or $(1-x)dx = \frac{1}{2}du$. The limits remain 0 to 1. Thus:

$$\int_0^1 (1-x)[x(2-x)]^K dx = \int_0^1 u^K \frac{1}{2}du = \frac{1}{2}\left[\frac{u^{K+1}}{K+1}\right]_0^1 = \frac{1}{2(K+1)}.$$

Substituting $K+1 = \alpha(\theta_1-\theta_2)+\theta_2$ back into the expression for I:

$$I = 2\theta_1^{\alpha}\theta_2^{1-\alpha} \cdot \frac{1}{2(\alpha(\theta_1-\theta_2)+\theta_2)} = \frac{\theta_1^{\alpha}\theta_2^{1-\alpha}}{\alpha(\theta_1-\theta_2)+\theta_2}.$$

Finally, taking the logarithm and multiplying by $\frac{1}{\alpha-1}$, we obtain Equation (30). □

3.3 Bhattacharyya Divergence

The Bhattacharyya divergence introduced by is a measure of distance between two probability distributions, closely related to the Bhattacharyya coefficient, which quantifies the overlap between two distributions and it is defined as:

$$D_{\mathrm{B}}(P \parallel Q) = 1 - \int \sqrt{p(x)q(x)}dx. \tag{32}$$

Note that in some papers the Bhattacharyya divergence is wrongly defined by

$$D_{\mathrm{B}}(P \parallel Q) = \int \sqrt{p(x)q(x)}dx.$$

However this is **Bhattacharyya** coefficient and applies as a similarity measure. For more details see [18].

Proposition 7. Consider two continuous random variables, X_1 and X_2, following a TL distribution with parameters θ_1 and θ_2, respectively. Then, the Bhattacharyya divergence between the PDF's $f_{X_1}(x)$ and $f_{X_2}(x)$ is given by:

$$D_B(P\|Q) = 1 - \frac{2\sqrt{\theta_1\theta_2}}{\theta_1+\theta_2}. \tag{33}$$

Proof. We first compute the Bhattacharyya coefficient. Since $\sqrt{f_{X_1}(x)f_{X_2}(x)} = 2\sqrt{\theta_1\theta_2}\,(1-x)\,[x(2-x)]^{\frac{\theta_1+\theta_2}{2}-1}$, the substitution $u = x(2-x)$, $du = 2(1-x)dx$, gives

$$\int_0^1 \sqrt{f_{X_1}(x)f_{X_2}(x)}\,dx = \sqrt{\theta_1\theta_2}\int_0^1 u^{\frac{\theta_1+\theta_2}{2}-1}du = \frac{2\sqrt{\theta_1\theta_2}}{\theta_1+\theta_2}.$$

Substituting into Equation (32) yields Equation (33). □

Example 6. Let two continuous random variables, X_1 and X_2, following a TL distribution with parameters θ_1 and $\theta_2 = k\theta_1$, respectively. Then, the Bhattacharyya divergence between the PDF's $f_{X_1}(x)$ and $f_{X_2}(x)$ can be yields as:

$$D_B(P\|Q) = 1 - \frac{2\sqrt{k}}{k+1}. \tag{34}$$

3.4 Stress-strength model

The stress-strength model is a well-established framework in reliability analysis, commonly employed in disciplines such as engineering, physics, and materials science to evaluate the probability of system survival under applied stress. In this model, reliability denoted by $\mathbf{R}$ is defined as the probability that the strength of a component, X_1, exceeds the stress, X_2, applied to it:

$$\mathbf{R} = P(X_1 > X_2).$$

A system is considered to fail when the applied stress exceeds its inherent strength, and conversely, it operates reliably when the strength surpasses the stress. Thus, $\mathbf{R}$ serves as a critical indicator of system performance, especially in electrical, mechanical, and structural applications. [6]

Proposition 8. Let $X_1 \sim TL(\theta_1)$ represent the strength of a component and $X_2 \sim TL(\theta_2)$ represent the stress applied to it. Assuming X_1 and X_2 are independent, the stress-strength reliability, defined as $R = P(X_1 > X_2)$, is given by:

$$R = \frac{\theta_1}{\theta_1 + \theta_2}. \tag{35}$$

Proof. The reliability R is evaluated as:

$$R = \int_0^1 f_{X_1}(x;\theta_1) F_{X_2}(x;\theta_2) dx.$$

Substituting the PDF of the strength (X_1) and the CDF of the stress (X_2) into the Equation:

$$R = \int_0^1 \left[2\theta_1(1-x)(x(2-x))^{\theta_1 - 1}\right] \cdot \left[(x(2-x))^{\theta_2}\right] dx.$$

To solve this integral, we use the change of variable $u = x(2-x)$. Consequently, the differential is $du = 2(1-x)dx$. The limits of integration remain from 0 to 1. The integral transforms to:

$$\begin{aligned} R &= \int_0^1 \theta_1 (x(2-x))^{\theta_1 - 1} (x(2-x))^{\theta_2} \cdot [2(1-x)dx] \\ &= \theta_1 \int_0^1 u^{\theta_1 - 1} u^{\theta_2} du \\ &= \theta_1 \int_0^1 u^{\theta_1 + \theta_2 - 1} du. \end{aligned}$$

Solving the standard power integral yields:

$$R = \theta_1 \left[\frac{u^{\theta_1 + \theta_2}}{\theta_1 + \theta_2}\right]_0^1 = \frac{\theta_1}{\theta_1 + \theta_2}.$$

This result confirms that reliability increases as the strength parameter θ_1 increases relative to the stress parameter θ_2. □

Proposition 9. Let $X_1 \sim \mathrm{TL}(\theta_1)$ and $X_2 \sim \mathrm{TL}(\theta_2)$ be independent strength and stress variables, respectively. If $\theta_2 = k\theta_1$, for some scalar $k > 0$, then we would have

$$\mathbf{R}(k) = \frac{1}{1+k}. \tag{36}$$

So, the properties of the reliability function is given by:

1. $\mathbf{R}(k)$ is strictly decreasing and convex in $k \in (0, \infty)$.
2. $\lim_{k\to 0} \mathbf{R}(k) = 1$, and $\lim_{k\to\infty} \mathbf{R}(k) = 0$.
3. $\mathbf{R}(1) = 0.5$: reliability is 50% when stress and strength follow identical distributions.
4. The hazard ratio k can be recovered from observed $\mathbf{R}$ by $k = \frac{1}{\mathbf{R}} - 1$.

Proposition 10. Let $X_1 \sim \mathrm{TL}(\theta_1)$ and $X_2 \sim \mathrm{TL}(\theta_2)$ be independent strength and stress random variables. Assume:

$$\theta_2 = \theta_1^k, \quad t > 0.$$

Then the reliability function $\mathbf{R}$ becomes:

$$\mathbf{R}(k, \theta_1) = \frac{1}{1+\theta_1^{k-1}}. \tag{37}$$

We can get approximately the behavior of $\mathbf{R}$ as follow:

$$\begin{cases} \begin{cases} k \to \infty & \mathrm{R} \to 0, \\ k = 1 & \mathrm{R} = 0.5, \\ k \to 0^+ & \mathrm{R} \to \dfrac{\theta_1}{1+\theta_1}. \end{cases} \\ \\ k \to 0^+ \quad \textbf{implies} \quad \begin{cases} \theta_1 \to 0^+ & \mathrm{R} \to 0^+, \\ \theta_1 = 1 & \mathrm{R} = 0.5, \\ \theta_1 \to \infty & \mathrm{R} \to 1. \end{cases} \\ \\ k \to \infty \quad \textbf{implies} \quad \begin{cases} \theta_1 \to 0^+ & \mathrm{R} \to 1, \\ \theta_1 = 1 & \mathrm{R} = 0.5, \\ \theta_1 \neq 1 & \mathrm{R} \to 0^+. \end{cases} \end{cases}$$

Example 7. Suppose the strength follows a TL distribution with parameter $\theta_1 = 2$, and we evaluate the reliability for various values of k such that $\theta_2 = k\theta_1$. Table 2 shows the results of $\mathbf{R}(k) = \frac{1}{1+k}$ and $\mathbf{R}(t, \theta_1)$.

3.5 Practical Implications for Reliability and Lifetime Analysis

The analytical results presented in this paper constitute a practical framework for reliability modeling. The stress–strength model derived in Subsection 3.4, specifically the expression $R = \frac{\theta_1}{\theta_1+\theta_2}$, provides a direct and efficient tool for designers to assess system reliability once the

Table 2: Computation of $\mathbf{R}(k)$ along with the corresponding values for k and θ_1.

k	$\mathbf{R}(\mathbf{k})$	$\theta_1 = 0.25$	$\theta_1 = 0.50$	$\theta_1 = 0.75$	$\theta_1 = 1$	$\theta_1 = 2$	$\theta_1 = 4$
0	1.000	0.500	0.500	0.500	0.500	0.500	0.500
0.5	0.667	0.667	0.586	0.536	0.500	0.414	0.333
1	0.500	0.800	0.667	0.571	0.500	0.333	0.200
2	0.333	0.941	0.800	0.640	0.500	0.200	0.059
5	0.167	0.999	0.970	0.808	0.500	0.030	0.001
10	0.091	1.000	0.999	0.947	0.500	0.001	0.000

stress and strength parameters are estimated. Similarly, the Shannon entropy and divergence measures derived in Section 3 can be employed to quantify the uncertainty and discrepancy between different reliability models or operating conditions, offering a systematic way to evaluate distributional sensitivity. The numerical validations and simulations provided throughout the paper serve to demonstrate the behavior of these measures, confirming their reliability and computational feasibility for practical use in reliability engineering and lifetime data analysis.

4 Conclusion

In this paper, we have provided an analytical treatment of the T–L distribution, covering key distributional properties, parameter estimation methods, and information-theoretic measures. The significance of this work lies in the derivation of exact analytical forms for essential statistical measures, such as the Shannon entropy, the Kullback–Leibler divergence, and stress–strength reliability indices. By reducing the reliance on numerical approximations, these results offer practitioners a more computationally efficient and precise framework for reliability modeling and decision-making under bounded lifetime scenarios.

Despite the contributions presented, this study has inherent limitations that provide scope for future research. First, the parameter estimation performance was evaluated using frequentist metrics (Bias and MSE) for the PWM and MLE methods. Future studies could investigate Bayesian estimation schemes, which might offer more robust performance, particularly in scenarios with limited sample sizes or near-boundary parameters. Second, the current work is limited to the univariate case. Extending the Topp–Leone framework to a bivariate or multivariate setting—perhaps through the use of copulas—remains an open and promising research avenue. Furthermore, while we have established the theoretical foundations and provided an initial assessment of the distribution's reliability applications, a comprehensive comparative analysis with other bounded lifetime models (such as the Beta, Kumaraswamy, or Unit-Weibull distributions) would be beneficial. Future work could also address the robustness of the derived measures against outliers or model misspecification. We believe that the analytical closed-forms derived herein will serve as a valuable reference for further developments in reliability engineering and related fields involving unit-interval data.

Declarations

Author Contributions

Conceptualization: Dr Alireza Pakgohar and Dr Soheil Shokri; Methodology: Pakgohar; Software and Validation: All authors; Formal Analysis: All authors; Investigation: All authors; Resources: Pakgohar and Shokri; Data Curation: Pakgohar; Writing – Original Draft: Pakgohar; Writing – Review & Editing: All authors; Visualization: Pakgohar; Supervision:Pakgohar; Project Administration: All authors; Funding Acquisition: None.

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Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have influenced the work reported in this paper.

Artificial Intelligence Statement

Artificial intelligence (AI) tools, including large language models, were used solely for language editing and improving readability. AI tools were not used for generating ideas, performing analyses, interpreting results, or writing the scientific content. All scientific conclusions and intellectual contributions were made exclusively by the authors.

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