Received: xxx **Accepted:** xxx **Published:** xxx.
DOI: 10.30473/coam.2026.75619.1332
Volume xxx, Issue xxx, p.p. 1-37: xxx
Research Article

Open Access

Control and Optimization in Applied Mathematics - COAM

Optimizing Cryptocurrency Investment Portfolios Under Uncertainty Using Network Data Envelopment Analysis, Robust Programming, and a Lagrangian Relaxation Algorithm

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How to Cite
Hamidieh, A., Akhgari Rik, B. (2027). "Optimizing cryptocurrency investment portfolios under uncertainty using network data envelopment analysis, robust programming, and a Lagrangian relaxation algorithm". *Control and Optimization in Applied Mathematics*, X(x), 1–37. https://doi.org/10.30473/coam.2026.75619.1332

Abstract. Optimizing investment portfolios by assessing efficiency in competitive environments is a vital topic in investment management. The volatility of financial markets and increasing uncertainty make it difficult to use traditional models effectively. As a result, there has been growing interest in optimizing financial models under uncertainty to better reflect real-world conditions. The present study develops a combined optimization model to evaluate risk under uncertainty in inputs and outputs. This model uses Network Data Envelopment Analysis (NDEA), robust optimization, and Lagrangian relaxation. The data for this study are the latest adjusted price data of 40 top cryptocurrencies from CoinMarketCap, used to estimate risk and create an optimal portfolio. A two-step approach is proposed to select and optimize the cryptocurrency portfolio, make the investment process more reliable, and comprehensively evaluate cryptocurrencies based on financial criteria. In the first step, the efficiency of the selected cryptocurrencies is evaluated using NDEA and robust programming. In the second step, Lagrangian relaxation is applied to handle the complicating constraints of the problem. This approach helps determine the selection of cryptocurrencies and the investment amounts, and provides a basis for allocating investment amounts. The computational results compare the direct solution of the model in GAMS with the Lagrangian relaxation algorithm on instances of increasing size.

Keywords. Cryptocurrency, Network data envelopment analysis, Robust programming, Lagrangian relaxation, Portfolio optimization.

MSC. 90C08; 90C17; 91G10.

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Corrected Proofs

1 Introduction

Technological advancements in recent decades have made digital cryptocurrencies dominant over physical cash in some countries [67]. Digital currencies enable individuals to transfer their money in a digital space. Cryptocurrency selection and optimization are key topics in investment management research. These issues have consistently attracted the attention of financial market researchers as an essential research area.

Investors or traders in financial markets seek profitability in the cryptocurrency market, aiming for better performance than other cryptocurrency portfolios in an economically and socially uncertain environment [60]. On the other hand, the diversity of cryptocurrencies introduces new challenges. In addition, the development of digital currencies has attracted the attention of many investors and policymakers [36, 55].

Decentralization is one feature of cryptocurrencies that has attracted attention. In other words, no group or organization controls them. Thus, transferring cryptocurrencies does not require banks or third parties; instead, they can be transferred directly to another person. These transfers are based on cryptographic systems such as blockchain [54], and investment decisions concerning blockchain technology have also been analyzed in operational settings [27]. Earning income from selected cryptocurrencies in an uncertain environment is difficult for investors. Hence, choosing a cryptocurrency portfolio to maximize profit and reduce risk is a concern for investors in financial markets. This concern is often marked by uncertain judgment and described by imprecise parameter values.

Regarding investment uncertainty, capital allocation to assets or risky portfolios is a primary concern for investors [24]. Therefore, understanding the connection between price changes in digital currencies can lead to potential profits for investors and prevent losses due to market fluctuations [37]. The risk criteria of financial portfolios include semi-variance, absolute deviations, and value at risk, based on which investors make decisions regarding trading or holding assets [43, 51]. In this respect, researchers have proposed various methods for analyzing the relationship between cryptocurrencies.

One strategy to address the uncertainty arising from the lack of knowledge, without considering the data distribution or complex constraints, is to employ robust optimization approaches, first developed by Soyster [61]. Although Soyster's interval-based approach provides full immunity against parameter fluctuations, it is widely recognized as being overly conservative [9]. To overcome this excessive conservatism, Ben-Tal and Nemirovski [9] introduced ellipsoidal uncertainty sets; however, their resulting robust counterpart formulations are generally nonlinear and computationally demanding. Subsequently, Bertsimas and Sim [10] developed the budgeted uncertainty framework, which preserves problem linearity while allowing decision-makers to flexibly control the degree of conservatism through the uncertainty budget Γ. More

Corrected proofs

recently, such robust optimization formulations have been successfully tailored and extended to portfolio selection and financial risk management problems [57, 69].

Therefore, uncertainty modeling is used to enhance the confidence level of the model's final response, based on uncertainty intervals and in the absence of precise input data. This modeling leads to the design of realistic systems in financial markets. In this regard, fuzzy, stochastic, and robust programming approaches are applied to optimize investment planning models [25, 66]. The main objective of the above models is to select cryptocurrencies with the best possible confidence level and meet the financial investor's expectations as closely as possible or within a stable range [15, 33]. Besides, the combination of currencies, identifying opportunities for evaluation, and aligning them with the goals and structure of an optimal cryptocurrency are also crucial when selecting cryptocurrencies [3]. The main issue is how to design a process for evaluating the efficiency of cryptocurrencies and selecting them in a way that ensures sufficient credibility. Such a process should also allow for allocating the investor's budget to the purchase of each cryptocurrency and making the final decision to implement the model to achieve an optimal cryptocurrency portfolio. Therefore, developing new cryptocurrency selection models that can work flexibly and comprehensively in the face of investor decisions under uncertain conditions has always been a challenging area of interest for researchers [19]. The critical issue in this respect is which model to choose for designing the portfolio in terms of currency type and the number of financial assets when dealing with uncertainty in key model parameters, ensuring minimal complexity. In addition, it is necessary to decide which approach to use to help the investor reduce decision-making risk with reliable and trustworthy planning. A powerful method for measuring and evaluating efficiency under uncertainty is the Network Data Envelopment Analysis (NDEA) model combined with robust programming [5, 17]. Such a model can be designed as a tool with flexible constraints for uncertain parameters in the problem space [10]. DEA is a non-parametric method in operations research and economics that estimates the efficiency of production or operational units. In this context, combined planning models are developed considering the practical goals and constraints in financial markets and better alignment with the real world.

In general, most previous studies in the field of data envelopment analysis (DEA) have used network models (NDEA) to examine the multi-stage structure of decision-making units; however, most of these models assume the data to be deterministic [39, 46]. In many real systems, including supply chains, manufacturing and service systems, the values of inputs, outputs and intermediate variables face some degree of uncertainty [5, 63]. In such situations, the use of deterministic models may lead to unstable or optimistic estimates of efficiency. On the other hand, although some studies have used the robust optimization approach to deal with uncertainty in DEA models, integrating this approach with the DEA network structure and providing an efficient solution framework has still received limited attention [5, 17].

Corrected proofs

The present study presents a hybrid optimization framework in which network data envelopment analysis is integrated with robust optimization to enable the evaluation of the performance of multi-stage systems under uncertainty. In addition, in order to control the computational complexity of the resulting model, the Lagrangian relaxation technique is used as a decomposition method that allows the transformation of the large problem into more solvable subproblems. Thus, the main innovation of this study is in providing an integrated framework that simultaneously models the network structure of processes, considers data uncertainty, and provides applicability to large-scale problems by utilizing an efficient solution method.

The main objective of the present research in portfolio management is to develop a decision-making tool that enables investors to select optimal assets under uncertainty. Therefore, a novel model is presented for more efficient cryptocurrency analysis, and a combined modeling approach is used to address uncertainty. In the first stage, the efficiency of investable cryptocurrencies is evaluated and measured. Subsequently, selected cryptocurrencies are passed through an investment filter and are identified as selected inputs. The allocated investment amount for each eligible selected cryptocurrency is determined in the next stage. Then, robust combined optimization methods and Lagrangian relaxation are applied to deal with parameter uncertainty. Finally, the developed research approach is tested on actual and state-of-the-art global cryptocurrency data. Overall, the present research aims to answer the following questions:

1. How does the robust NDEA model respond to uncertain conditions?
2. What is the impact of changes in the conservatism parameter of the robust model on cryptocurrency and portfolio evaluation results?
3. How do robust counterpart models perform when selecting the optimal portfolio?
4. Is the proposed approach effective in constructing a cryptocurrency portfolio in an uncertain environment?
5. How does the Lagrange multiplier affect the performance of the objective function?

The remainder of this research is structured as follows. Section 2 examines the theoretical foundations and reviews domestic and foreign studies. Section 3 covers the research methodology and elaborates on the proposed NDEA-based method. Subsequently, the model test and analysis of findings are presented in Section 4. Finally, Section 5 presents the discussion and conclusions.

2 Review of Research Background

To select a portfolio and adequately understand the risk characteristics of cryptocurrencies, investors must examine the main features of cryptocurrency returns, including distribution, lack of correlation, volatility clustering, leverage effect, and long-term dependence [41]. When choos-

Corrected proofs

ing cryptocurrencies, investors pay close attention to transactional data, such as the initial price, closing price, trading volume, lowest price, and highest price. These commercial data represent the performance of digital currencies and serve as a strategic guide for investors [13]. Pricing cryptocurrency options is a crucial factor in risk management and market stabilization. This factor presents unique challenges due to specific underlying dynamics, such as the reversal of the leverage effect. Classic option pricing models (e.g., Black–Scholes and Heston) struggle to address these dynamics due to their set assumptions [14]. GARCH-type models have also been used to examine volatility spillovers within the cryptocurrency market; for instance, Katsiampa et al. [42] reported bi-directional volatility spillovers among Bitcoin, Ether, and Litecoin. In the Fourth Industrial Revolution era, cryptocurrencies emerged as technological and financial products and a global phenomenon with profound potential for social welfare while introducing new risks and uncertainties. Umar et al. [62] examined the presence, nature, and extent of risk-return and volatility spillovers originating from the cryptocurrency market and transferring to the global financial system.

In financial markets and investment, we encounter many uncertain parameters. As a result, robust optimization approaches are used to deal with uncertain data, first developed by Soyster [61]. Soyster's approach produced feasible and near-optimal solutions in the face of imprecise input parameters. However, it was highly conservative, leading to a significant gap from optimality under nominal values [9]. The robust optimization approach by Ben-Tal and Nemirovski [9] was developed for problems with ellipsoidal uncertainty sets, providing a new robust counterpart for linear programming problems. Although it had a lower conservatism level than Soyster's model, the final model was nonlinear and not applicable to discrete optimization models. In this context, Bertsimas and Sim introduced a different method to tune conservatism [10]. This method led to a linear optimization model that could be implemented in discrete models. Consequently, robust portfolio optimization models were developed based on the ability to adjust the level of conservatism in the robust solution [32, 49, 57]. However, numerous studies have been conducted to develop investment models under uncertainty, seeking to optimize investment constraints and improve the investment process performance.

Burkaltseva et al. (2021) [16] examined the methodological foundations of stock market risk in developed and developing countries during crises. They also explored the development of stock market growth methods through structural changes in the economic development system and advances in digital technologies. Goel et al. (2018) [31] used a mixed Conditional Value at Risk (CVaR) model to create an optimal index-tracking portfolio. They proposed an investment strategy of passive management that follows an index or a portfolio with market-weighted assets. The goal of this management style is to minimize investment fees and avoid the adverse outcomes of incorrect future predictions. On the other hand, active portfolio management aims to outperform index returns, while passive management seeks to achieve returns and risks that match the index. Moreover, Peykani et al. (2020) [57] developed a two-phase

Corrected proofs

robust portfolio selection and optimization approach under uncertainty, applied to the Tehran Stock Exchange. Larni-Fooeik et al. (2024) [44] highlighted that portfolio optimization involves finding the best combination of securities and stocks to reduce risk and increase profit. To assess the impact of risk on portfolio optimization, they considered volatility risk measures. They used stochastic optimization to identify the most suitable scenario that not only maximizes returns but also minimizes regret. Also, to transform a multi-objective model into a single objective, they used the augmented epsilon constraint method (AEC) to create the Pareto efficiency frontier.

Arabmaldar et al. (2017) [5] proposed a new robust DEA model and a super-efficiency measure. Chuliá et al. (2017) [21] proposed a measure of uncertainty in the stock market. Doshi et al. (2018) [24] examined how uncertainty affects corporate capital investment and risk-management decisions. Escobari and Jafarinejad (2019) [26] investigated investor uncertainty and stock market risk. To this end, they proposed a new technique to model investment uncertainty using the conditional volatility of investor sentiment. They worked with weekly data on investor sentiment, major US stock indices, and alternative uncertainty measures and conducted several tests to validate the proposed measure. The estimates showed that investor uncertainty is higher during economic recessions and is associated with lower investor sentiment. Additionally, the results supported a positive correlation between sentiment and returns, which was interpreted as a positive link between investor uncertainty and market risk. Ahn et al. (2019) [2] examined the effects of stock market uncertainty on economic fundamentals in China. They used entropy measures and symbolic time-series analysis to estimate uncertainty in the market. The findings revealed that stock market uncertainty significantly affects economic activity and systemic risk, and that an uncertainty shock leads to a short-term reduction in industrial production. Zhu and Zhang (2021) [69] studied long-term wealth growth portfolio allocation under parameter uncertainty using a non-conservative robust approach. Luan et al. (2021) [49] proposed a two-stage robust optimization model for investment portfolio problems under uncertainty. Li and Teo (2021) [45] explored portfolio optimization in real financial markets, addressing both uncertainty and randomness simultaneously. They first derived skewness for three types of uncertain random variables. Then, considering risk preferences in an uncertain stochastic environment, they proposed a mean-variance-skewness model for portfolio optimization. Avellán et al. (2022) [7] developed a streamlined approach to measuring uncertainty and applied it to the Ecuadorian economy. Chou and Chen (2024) [20] used decision tree-based and metaheuristic optimization algorithms to create an efficient stock selection framework based on financial analysis. This approach provides a profitable investment strategy integrated with portfolio optimization while systematically lowering the portfolio's investment threshold.

Ahir et al. (2022) [1] examined the Global Uncertainty Index by creating the World Uncertainty Index (WUI) for 143 countries independently and seasonally. This index rises globally

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in response to significant events such as the Gulf War, the Euro crisis, the Brexit vote, and the COVID-19 pandemic. Uncertainty levels are higher in developing countries, but the trend is more synchronized and compact among advanced economies with trade and financial ties. Regression estimates showed that financial and investment innovations significantly reduced WUI levels. Table 1 presents a more detailed classification of the literature to highlight the gaps and the innovations of the developed approach.

Based on the research conducted, the presence of uncertainty in data due to forecasting, measurement, or implementation errors is addressed using stochastic and robust approaches. Unlike stochastic models, robust approaches have high flexibility and efficiency, which is why they have been developed in recent research.

After reviewing the existing studies, the innovations of this research are as follows:

- Implementing an efficiency-based portfolio optimization approach in a new case study (cryptocurrency market), evaluating various financial aspects and criteria in an uncertain market environment, and considering actual uncertainty conditions in investment amounts and cryptocurrency selection;
- Applying the Lagrangian relaxation algorithm to solve complex model constraints;
- Implementing a two-stage NDEA model to improve the problem solution;
- Using a realistic, robust programming approach to increase the level of risk aversion in the problem;
- Including fees in the objective function (which has not been considered in previous research);
- Securing the optimal solutions to the problem under the most unfavorable conditions.

As mentioned, robust optimization and data envelopment analysis approaches have been considered separately in previous literature in studies related to cryptocurrency portfolio selection. However, a literature review shows that not only has network data envelopment analysis not been directly used in the above research, but a coherent methodological link between these areas has not yet been formed. Most of the research on digital asset portfolios has used classical portfolio optimization models or statistical and machine learning approaches, and has not focused on evaluating the multi-stage structure of asset performance. On the other hand, DEA and NDEA studies have mainly focused on evaluating the efficiency of decision-making units in areas such as banking, manufacturing, and supply chains, and their application to digital assets and investment portfolio management is still limited. In addition, many existing DEA models assume deterministic data, while the digital asset market is inherently volatile and uncertain. As a result, deterministic models may lead to assessments that are sensitive to data

Corrected proofs

Table 1: Summary of the literature.

Study	Model / method	Uncertainty	Assessment	NDEA structure	Data type	Issue
Brini and Lenz (2024) [14]	Machine learning regression	No	Pricing	One-step	Deterministic	Pricing cryptocurrency options
Chou and Chen (2024) [20]	Tree-based models and the FBI algorithm	No	Investment amount; Currency selection	One-step	Deterministic	Optimizing investment portfolios
Amirteimoori (2024) [4]	Two-Stage DEA Model	No	Performance of investment companies	Network; Two-step	Deterministic	Banking sector
Jiang et al. (2024) [37]	Quantile regression approach	No	Investment amount; Performance of investment companies	One-step	Deterministic	Cryptocurrency market and carbon market
Fukuyama et al. (2023) [30]	Dynamic NDEA	No	Performance of investment companies	Network	Deterministic	Chinese banking industry
Ma et al. (2023) [50]	Neural networks	Yes	Investment amount; Performance of investment companies	Network	Deterministic	Portfolios performance
Lotfi et al. (2023) [48]	Robust optimization and stochastic programming	Yes	Performance of investment companies	Two-step	–	Optimizing cryptocurrency portfolio
Oyewola et al. (2022) [56]	Statistical models, machine learning algorithms	Yes	Performance of investment companies	One-step	–	Optimizing cryptocurrency prediction
Gong et al. (2021) [34]	Fuzzy multi-objective portfolio model involving DEA	Yes	Currency selection	One-step	Uncertain inputs	Cross-efficiency
Chang et al. (2021) [17]	Markov chain Monte Carlo Bayesian algorithms	No	Performance of investment companies	Network	Deterministic	Portfolio evaluation
Burggraf (2021) [15]	Machine learning-based hierarchical risk parity approach- Robust	Yes	Performance of investment companies	One-step	–	Optimizing investment portfolios
Ahn et al. (2019) [2]	Entropy-based approach	No	Investment amount	One-step	–	Stock market Uncertainty
Goel et al. (2018) [31]	Value-at-risk	Yes	Investment amount; Performance of investment companies	Two-step	–	Index tracking
Doshi et al. (2018) [24]	Regression	Yes	Pricing	Two-step	–	Capital investment, and risk management
Ranković et al. (2016) [58]	Genetic algorithm	Yes	Investment amount	One-step	–	Mean-univariate GARCH VaR portfolio optimization
Babaei et al. (2015) [8]	Particle swarm optimization (MOPSO) algorithms	Yes	Investment amount; Performance of investment companies	–	–	Multi-objective portfolio optimization
Kakouris and Rustem (2014) [38]	A Robust programming and CVaR approach	Yes	Investment amount; Performance of investment companies	–	–	Robust portfolio optimization
Present study	NDEA - robust and Lagrangian Relaxation	Yes	Investment amount; Currency selection	Network	Uncertain inputs and outputs	Optimizing the cryptocurrency investment portfolio

Corrected proofs

changes. The present study addresses this gap by providing an integrated framework. In this study, first, the network data envelopment analysis model is used to model the multi-stage structure of cryptocurrency performance, and then, in order to consider the uncertainty in the market data, a robust optimization framework is introduced into the model. In addition, to control the computational complexity of the resulting model, the Lagrangian relaxation technique is used as an efficient solution method. Thus, by combining these three approaches, this study provides a framework that can evaluate the efficiency and select the cryptocurrency portfolio more realistically and stably under market uncertainty. This topic covers the gap between network efficiency evaluation methods, uncertainty management, and their application in the field of cryptocurrency.

3 Research Methodology

In this study, a combined Data Envelopment Analysis (DEA)–robust planning approach is developed under uncertainty conditions for cryptocurrency portfolio optimization. In the first stage, the efficiency of investable cryptocurrencies is measured and evaluated using the NDEA approach. This stage involves filtering cryptocurrencies to begin investment in the second phase; next, the investment amount in the qualified cryptocurrencies is determined; finally, the selected portfolio is created. In each phase, robust planning addresses parameter uncertainty. Finally, the research pathway is implemented through a cryptocurrency case study. The Lagrangian relaxation algorithm is then used to improve model efficiency and performance. This approach consists of two phases. Figure 1 shows the overall structure of the research.

3.1 Portfolio Models and Classical Risk Measures

The first method for portfolio selection was proposed by Markowitz [51]. The Mean-Variance (MV) model for solving the portfolio selection problem is as follows. The notation used in this subsection is: $j, h = 1, \ldots, n$ index the cryptocurrencies; Θ_{MV} denotes the portfolio weight vector of the MV model; $\bar{R}_j$ is the mean return of cryptocurrency j; σ_j^2 is the variance of the return of cryptocurrency j; σ_{jh} is the covariance between cryptocurrencies j and h; R_E is the expected return measure for the portfolio; and ω_j is the weight of cryptocurrency j in the portfolio.

Corrected proofs

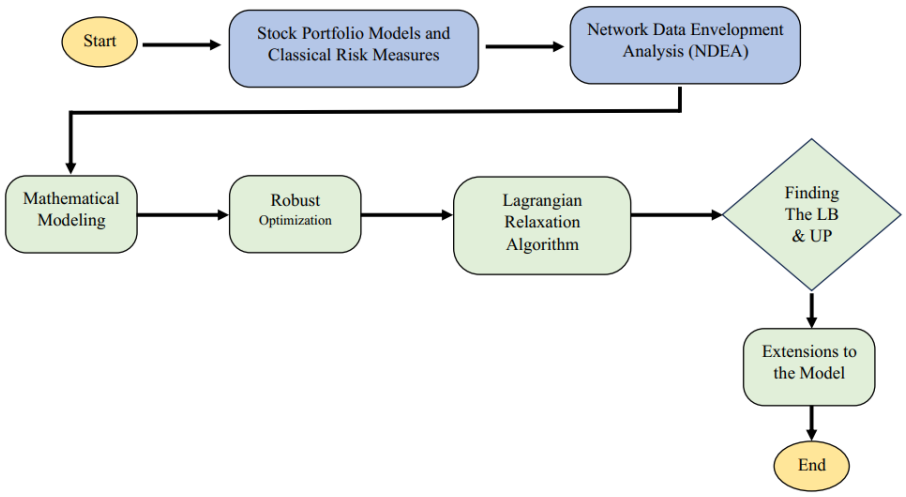


Figure 1: The overall structure of the present research.

$$\begin{aligned}
\Theta_{\text{MV}}: \quad \min \ & \sum_{j=1}^{n} \omega_j^2 \sigma_j^2 + 2 \sum_{j=1}^{n-1} \sum_{h=j+1}^{n} \omega_j \omega_h \sigma_{jh} = \sum_{j=1}^{n} \sum_{h=1}^{n} \omega_j \omega_h \sigma_{jh} \\
\text{s.t.} \ & \sum_{j=1}^{n} \bar{R}_j \, \omega_j \geq R_E, \\
& \sum_{j=1}^{n} \omega_j = 1, \\
& \omega_j \geq 0 \quad \forall j.
\end{aligned} \tag{1}$$

As shown in Model (1), variance is used as the risk measure for the portfolio. Note that using variance as a risk measure for portfolio selection penalizes both high and low expected returns. Markowitz [52] proposed semi-variance as a downside risk measure, focusing on the probability of returns falling below the expected return. The definition of the semi-variance risk measure is given in Equation (2):

$$\begin{aligned}
\text{SV} &= E\left(\left(\max\left\{ 0, \ R_E - \sum_{j=1}^{n} R_{ij} \omega_j \right\} \right)^2 \right) \\
&= \begin{cases} \left(R_E - \sum_{j=1}^{n} R_{ij} \omega_j \right)^2, & R_E - \sum_{j=1}^{n} R_{ij} \omega_j > 0, \\ 0, & R_E - \sum_{j=1}^{n} R_{ij} \omega_j \leq 0. \end{cases}
\end{aligned} \tag{2}$$

Decision-makers need a covariance matrix when solving the mean-variance model. Although it is difficult to estimate this matrix with real data, there is no need to calculate the

Corrected Proofs

covariance matrix when using the Mean Semi-Variance (MSV) model; instead, the joint distribution of returns must be calculated.

Since the original Markowitz model [51] is a Quadratic Programming (QP) model and is challenging to solve for large datasets, Konno and Yamazaki [43] proposed using Absolute Deviation (AD) instead of variance as a risk measure for portfolio selection. The Mean Absolute Deviation (MAD) model is a Linear Programming (LP) model that reduces computation time. The definition of absolute deviation is given by Equation (3):

$$\mathrm{AD} = \left| R_E - \sum_{j=1}^{n} R_{ij}\omega_j \right| = \begin{cases} R_E - \sum_{j=1}^{n} R_{ij}\omega_j, & R_E - \sum_{j=1}^{n} R_{ij}\omega_j > 0, \\ \sum_{j=1}^{n} R_{ij}\omega_j - R_E, & R_E - \sum_{j=1}^{n} R_{ij}\omega_j \leq 0. \end{cases} \tag{3}$$

This risk measure quantifies the deviation from expected returns. Meanwhile, with the MAD model, there is no need to calculate the covariance matrix.

3.2 Network Data Envelopment Analysis (NDEA)

Data Envelopment Analysis (DEA) is a technique used to evaluate the performance of a set of similar decision-making units. It is a non-parametric approach for measuring the relative efficiency of independent decision-making units. The model has multiple inputs and outputs without imposing any specific form on the relationships between inputs and outputs [12, 22]. In the classical DEA model of Charnes et al. [18], which follows a "black-box" approach, only the inputs and outputs are considered in the evaluation without considering the internal structure. As a result, the source of inefficiency in non-efficient units is not identified. To address this challenge, the Network Data Envelopment Analysis (NDEA) model [23] was developed to examine the operations of processes and components in evaluating system efficiency [39].

Investment performance in the capital market is a multi-stage process that classical models (black-box models) cannot fully evaluate. However, NDEA models define production and intermediary structures for investment inputs and outputs [47]. In the production approach, investment deposits are considered as outputs, while they are treated as inputs in the intermediary approach. In the NDEA model, investment deposits are outputs at one stage and inputs at the next stage. Moreover, since capital market activities occur over consecutive periods and are interconnected, the performance of the capital market is evaluated over time, and the connections between stages are considered through intermediate variables [17]. Also, models that consider internal structure and time are used for capital market evaluation. Therefore, this study considers the structure of the cryptocurrency market, the efficiency of each stage, and the network efficiency over the relevant period. Figure 2 shows the Two-Stage NDEA Framework.

In previous studies, the single-stage DEA (black-box model) was used to calculate the efficiency of each stage and the overall efficiency [22, 64]. The black-box model is a baseline for

Corrected proofs



Figure 2: Two-Stage NDEA.

comparing and analyzing efficiency. This model, in its multiplier form, is given by

$$\begin{aligned} \max \ & \sum_{q=1}^{Q} w_{qk}\, y_{qk} \\ \text{s.t.} \ & \sum_{i=1}^{I^1} V_{ik}^{(1)} X_{ik}^{(1)} + \sum_{i=I^1+1}^{I^2} V_{ik}^{(2)} X_{ik}^{(2)} = 1, \\ & \sum_{q=1}^{Q} w_{qk}\, y_{qj} - \sum_{i=1}^{I^1} V_{ik}^{(1)} X_{ij}^{(1)} - \sum_{i=I^1+1}^{I^2} V_{ik}^{(2)} X_{ij}^{(2)} \le 0, \\ & V_{ik}^{(1)} \ge \varepsilon,\ V_{ik}^{(2)} \ge \varepsilon,\ w_{qk} \ge \varepsilon. \end{aligned} \tag{4}$$

Here $j = 1, \ldots, n$ indexes the DMUs (cryptocurrencies) and k denotes the DMU under evaluation.

Model (4) involves a production stage where inputs (X) are converted into desired outputs (y). The parameters w_{qk} and V_{ik} are the multipliers (weights) of the q-th desired output and of the i-th input, respectively, and ε is a small non-Archimedean number. In this model, the internal structure of the DMUs is not examined, and the conflict between the two stages due to intermediate components is not resolved. In other words, the internal financial structure is overlooked [22, **?**]. This study uses a two-stage production platform: in the first stage, inputs are converted into intermediate products (overall financing scale). In the second stage, the desired output is measured with the intermediate product [39]. Traditional DEA models treat each decision-making unit as a "black box", ignoring the internal interactions among sub-processes. To open this black box, network DEA formulations were pioneered by Färe and Grosskopf [28] and comprehensively systematized by Cook and Zhu [23]; two-stage structures with intermediate products were also analyzed by Kao and Hwang [40]. Accordingly, the overall efficiency of the two-stage network, where its subprocesses are sequentially connected, is calculated based on the efficiency of the subprocesses and the scope of financing [23]. Based on the models by Zhang et al. [68], Kao [39], Cook and Zhu [23], and Cook and Seiford [22], the primary two-stage network DEA model is formulated as follows:

Corrected proofs

$$
\begin{aligned}
\max \ & \sum_{q=1}^{Q} w_{qk}\, y_{qk} \\
\text{s.t.} \ & \sum_{i=1}^{I^1} V_{ik}^{(1)} X_{ik}^{(1)} + \sum_{i=I^1+1}^{I^2} V_{ik}^{(2)} X_{ik}^{(2)} = 1, \\
& \sum_{n=1}^{N} m_{nk}\, z_{nj} - \sum_{i=1}^{I^1} V_{ik}^{(1)} X_{ij}^{(1)} \le 0, \\
& \sum_{q=1}^{Q} w_{qk}\, y_{qj} - \sum_{i=I^1+1}^{I^2} V_{ik}^{(2)} X_{ij}^{(2)} - \sum_{n=1}^{N} m_{nk}\, z_{nj} \le 0, \\
& V_{ik}^{(1)} \ge \varepsilon,\ V_{ik}^{(2)} \ge \varepsilon,\ w_{qk} \ge \varepsilon,\ m_{nk} \ge \varepsilon.
\end{aligned}
\tag{5}
$$

In Model (5), z_{nj} denotes the n-th intermediate product of DMU j and m_{nk} is the multiplier of the n-th intermediate product. The index j runs over all DMUs and k denotes the DMU under evaluation. After solving Model (5), the overall efficiency (EM_k), the stage 1 efficiency ($\mathrm{EM}_k^{(1)}$), and the stage 2 efficiency ($\mathrm{EM}_k^{(2)}$) are obtained using

$$
\begin{aligned}
\mathrm{EM}_k &= \frac{\sum_{q=1}^{Q} w_{qk}^{*}\, y_{qj}}{\sum_{i=1}^{I^1} V_{ik}^{(1)*} X_{ij}^{(1)} + \sum_{i=I^1+1}^{I^2} V_{ik}^{(2)*} X_{ij}^{(2)}}, \\
\mathrm{EM}_k^{(1)} &= \frac{\sum_{n=1}^{N} m_{nk}^{*}\, z_{nj}}{\sum_{i=1}^{I^1} V_{ik}^{(1)*} X_{ij}^{(1)}}, \\
\mathrm{EM}_k^{(2)} &= \frac{\sum_{q=1}^{Q} w_{qk}^{*}\, y_{qj}}{\sum_{i=I^1+1}^{I^2} V_{ik}^{(2)*} X_{ij}^{(2)} + \sum_{n=1}^{N} m_{nk}^{*}\, z_{nj}}.
\end{aligned}
\tag{6}
$$

Also, in the two-stage NDEA framework, the total efficiency is not necessarily equal to the simple product of the stage efficiencies $\mathrm{EM}_k^{(1)}$ and $\mathrm{EM}_k^{(2)}$ and (depending on the structure of the second-stage inputs) can be expressed as an algebraic combination of the stage efficiencies. Using the optimal weights obtained from solving Model (5), the total efficiency, the first-stage efficiency, and the second-stage efficiency are calculated according to Equation (6), and the relationship between them for DMU_j is as follows (ρ_k is the share of the stage 2-specific inputs in the total inputs):

$$
\begin{aligned}
\mathrm{EM}_k &= \rho_k\, \mathrm{EM}_k^{(2)} + (1-\rho_k)\big(\mathrm{EM}_k^{(1)} \mathrm{EM}_k^{(2)}\big), \\
\rho_k &= \frac{\sum_i V_{ik}^{(2)} X_{ij}^{(2)}}{\sum_i V_{ik}^{(1)} X_{ij}^{(1)} + \sum_i V_{ik}^{(2)} X_{ij}^{(2)}}, \\
& 0 \le \rho_k \le 1.
\end{aligned}
\tag{7}
$$

Corrected proofs

The data used in this study are classified into three categories, including input, intermediate, and output indicators. It is assumed that reducing the values of input indicators improves performance evaluation, while output indicators increase values. Intermediate indicators are the outputs of the first stage and the inputs of the second stage. Additionally, returns to scale are considered constant.

Specifically, the selection of variables in the network data envelopment analysis model is guided by two fundamental principles: (1) theoretical grounding in the financial and asset performance evaluation literature, and (2) practical relevance established in prior empirical DEA and portfolio management studies. In the financial literature, asset performance evaluation is usually based on the relationship between risk, return, and liquidity characteristics. Accordingly, the input indicators in this study represent the sources or risk factors that the investor incurs to achieve returns; the intermediate indicators show the process of converting risk and market characteristics into investment performance; and the output indicators represent the final results of asset performance, especially from the perspective of investment return and efficiency. In addition to the theoretical foundations, the selection of these indicators was informed by previous empirical studies in DEA and financial asset performance evaluation, in which variables such as risk, liquidity, and return measures have been widely used to analyze asset efficiency. In the NDEA framework, the use of intermediate variables allows modeling the different stages of value creation. It helps to more accurately understand the mechanism of converting risk and market characteristics into final returns. Therefore, the indicators used in this study are not only consistent with the theoretical foundations of financial performance evaluation, but also with common approaches in DEA/NDEA studies.

The input, intermediate, and output indicators are presented in Table 2. Accordingly, the research model is developed based on the models by Zhang et al. [68], Kao [39], and Cook and Seiford [22], which are further detailed below.

The linear form of NDEA is as follows.

$$\max\ c_s\Big(\sum_i\sum_j\sum_t v_{it}VD_{ijt}\Big) - c_b\Big(\sum_r\sum_j\sum_t Xo_{rj(t+1)}u_{rt}\Big) \tag{8}$$

$$\text{s.t.}\ \sum_r\sum_j\sum_t Xo_{rj(t+1)}u_{rt} + \sum_j\sum_h\sum_t ym_{hj(t+1)}vm_{ht} = 1 \tag{9}$$

$$\sum_r\sum_t u_{rt}B_{jt} + \sum_h\sum_t ym_{hj(t+1)}vm_{ht} \le \sum_t \omega_{jt}A_{jt} \qquad \forall j \tag{10}$$

$$\sum_r\sum_t Xo_{rj(t+1)}u_{rt} + \sum_h\sum_t vm_{ht}ym_{hj(t+1)} \le \sum_i\sum_t VD_{ijt}v_{it} + \sum_t A_{jt}\omega_{jt} \qquad \forall j \tag{11}$$

$$u_{rt},\ vm_{ht},\ \omega_{jt},\ v_{it} \ge \varepsilon > 0$$

Corrected proofs

Table 2: Input, intermediate, and output indicators of the NDEA model for the cryptocurrency portfolio.

Category	Financial criterion	Symbol	Description
Input	Network Value to Transactions Ratio (NVT)	$I^{(1)}$	Market value to transfer volume ratio
	Stock-to-Flow Ratio	$I^{(2)}$	Ratio between current asset holdings and new production flow
	Standard Deviation	$I^{(3)}$	Total risk (volatility of returns)
	Relative Strength Index (RSI)	$I^{(4)}$	Momentum oscillator measuring the speed and magnitude of recent price changes
Intermediate	Quick Ratio	$M^{(1)}$	Immediate liquidity of the asset in the market, defined as current assets minus inventory, divided by current liabilities: $(\mathrm{CA} - \mathrm{Inv})/\mathrm{CL}$
	Earnings Growth Rate per Cryptocurrency	$M^{(2)}$	Average network profit of the current three periods divided by the average network profit of the previous three periods
Output	Earnings per Cryptocurrency	$O^{(1)}$	Net income minus dividends on common stock
	Rate of Return	$O^{(2)}$	Profit or loss ratio of an investment in a given period
	On-Balance Volume (OBV)	$O^{(3)}$	Cryptocurrency momentum
	Average Directional Index (ADX)	$O^{(4)}$	Trend strength

Corrected proofs

3.3 Mathematical Model

To relate the described problem, the indices, parameters, and variables are defined below, and the linear form of the NDEA model is presented.

Indices	
A set of cryptocurrencies, $j = 1, \ldots, n$	j
A set of inputs, $i = 1, \ldots, m$	i
A set of desirable outputs, $r = 1, \ldots, R$	r
A set of desirable intermediate levels, $h = 1, \ldots, H$	h
A set of periods, $t = 1, \ldots, T$	t
Parameters	
The return of cryptocurrency j in period t	R_{jt}
The measure or target level of the expected return of the cryptocurrency	R_E
The mean return of cryptocurrency j	$\bar{R}_j$
The variance of cryptocurrency j	σ_j^2
The covariance between cryptocurrencies j and h	σ_{jh}
The i-th input vector of cryptocurrency j in period t	VD_{ijt}
The minimum total amount of funds available for investment in cryptocurrency j in period t	A_{jt}
The maximum total amount of funds available for investment in cryptocurrency j in period t	B_{jt}
The r-th desirable output level vector of cryptocurrency j from period $t+1$	$Xo_{rj(t+1)}$
The value of the h-th intermediate measure for cryptocurrency j in period $t+1$	$ym_{hj(t+1)}$
The level of conservatism (uncertainty budget)	Γ
Disturbance in uncertain parameters	Δ
Purchase fee	c_b
Sale fee	c_s
The amount of investment in cryptocurrency j in period t	In_{jt}
Decision variables	
The weight of cryptocurrency j in the cryptocurrency portfolio in period t	ω_{jt}
The enhancement factor for the r-th desirable output in period t	u_{rt}
The weight for the i-th input in period t	v_{it}
The enhancement factor for the h-th desirable intermediate level in period t	vm_{ht}
Supply shortfall of cryptocurrency j in period t	DT_{jt}

Equation (8) represents the main objective function, in which the sale-fee-weighted virtual input value is maximized net of the purchase-fee-weighted virtual output value. Constraint (9)

Corrected proofs

ensures that the weighted sum of the system outputs for each decision maker is normalized to unity (1). This not only prevents the model from assigning unlimited weights, but also allows for systematic comparisons between time periods (t) and different paths (r, j), and transforms the model from nonlinear to linear. Constraint (10) states that the total outputs for all decision-makers must be less than the inputs. Constraint (11) indicates that the total investable funds and inputs must be greater than the desired outputs.

In the present model, the parameter $\varepsilon > 0$ is defined as an infinitesimal non-Archimedean value (taken as a small value, such as $\varepsilon = 10^{-6}$, in computational and software implementations). Using the condition $u_{rt}, vm_{ht}, \omega_{jt}, v_{it} \geq \varepsilon > 0$ instead of the strict condition "greater than zero" not only prevents decision-making units from assigning zero weight to indicators but also leads to a closed feasible region for the problem. From a mathematical analysis perspective, the closure of this region, together with the boundedness of the objective function over it, guarantees the existence of an optimal point corresponding to the maximum operator (max) and prevents theoretical ambiguity related to the existence of a supremum (sup) in open domains [59].

3.4 Robust Optimization

Robust optimization is a risk-averse technique to handle optimization problems under uncertainty, aiming to achieve two important goals: model robustness and solution robustness [53]. Model robustness means that the solution remains feasible for nearly all possible values of uncertain parameters. Meanwhile, solution robustness means that the objective function value exhibits minimal undesirable deviations from the optimal value under uncertainty.

The well-known models in the field of robust optimization are those by Soyster [61], Ben-Tal and Nemirovski [9], and Bertsimas and Sim [10], each with its strengths and weaknesses. The main issue with Soyster's method is that it does not consider the entire objective function to ensure the robustness of the solution. Ben-Tal's model allows the decision-maker to specify the confidence level. However, a major drawback of this model is that it changes the complexity of the problem. In Bertsimas and Sim's model, not only can the confidence level be determined within the constraints, but the structure of the programming model is also preserved. Therefore, Bertsimas and Sim's method [10] is used to fulfill the model's robustness in this article.

The Bertsimas and Sim robust method is presented with the following characteristics. Consider the following linear optimization problem:

$$\begin{aligned} &\max\ CX \\ &\text{s.t.}\ \sum_{j} \tilde{\theta}_{ij} x_j \leq b_i \quad \forall i, \\ &\quad x_j \geq 0, \end{aligned} \tag{12}$$

Corrected proofs

where the parameters $\tilde{\theta}_{ij}$ are uncertain technical coefficients, and b_i is the right-hand-side parameter of constraint i. It is assumed that uncertainty in the constraints is independent. The uncertain parameter is defined as follows:

$$\tilde{\theta}_{ij} = \theta_{ij} + \mu_{ij}\hat{\theta}_{ij}, \tag{13}$$

where θ_{ij} are the nominal values, $\hat{\theta}_{ij}$ is a specified percentage of the nominal value for uncertain parameters, and μ_{ij} represents independent, symmetrically distributed random variables bounded in the interval $[-1, 1]$. The robust form of the above model using the Bertsimas and Sim approach is as follows:

$$\begin{aligned} &\max\ CX \\ &\text{s.t.}\ \sum_j \theta_{ij}x_j + \sum_j p_{ij} + z_i\Gamma \le b_i \quad \forall i, \\ &\quad z_i + p_{ij} \ge \hat{\theta}_{ij}x_j \quad \forall j, i, \\ &\quad z_i,\ p_{ij},\ x_j \ge 0 \quad \forall j, \forall i, \end{aligned} \tag{14}$$

where Γ determines the conservatism level in each constraint. If $\Gamma = 0$, there is no conservatism against deviations in the data. Also, if Γ equals the number of uncertain data points in the system, the highest level of conservatism is considered.

Based on the above relationships, this article further develops robust NDEA models with simultaneously uncertain inputs and outputs. For cases where uncertainty exists in all variables, modifications to the model are necessary. Model (15) is the base CCR model. The basis for developing NDEA models is defined by simultaneous uncertainty in inputs and outputs. The objective function of the model is a fractional efficiency index that represents the ratio of aggregated desired outputs to the corresponding inputs/costs; hence, a larger value means a better relative performance of the decision-making unit. Under the standard DEA assumptions, including non-negativity of the weight coefficients and the existence of a normalization constraint, the model is feasible if the data are non-degenerate and there is at least one permissible weight set. Also, due to the normalization constraint and the positivity of the denominator, the objective function is bounded from above and therefore has a finite optimal value. In the CCR framework, this index is usually interpreted as a measure of relative efficiency and, depending on the type of normalization, falls within a finite interval.

$$\begin{aligned} &\max\ \frac{\sum_i \sum_j \sum_t c_s\left(v_{it} VD_{ijt}\right)}{\sum_t \sum_i \sum_j Xo_{rj(t+1)} x_i} \\ &\text{s.t.}\ \frac{\sum_i \sum_t c_s\left(v_{it} VD_{ijt}\right)}{\sum_t \sum_r Xo_{rj(t+1)} x_{ij}} \le 1 \quad \forall j, \\ &\quad v_{it},\ Xo_{rj(t+1)} \ge 0. \end{aligned} \tag{15}$$

Corrected proofs

In the above model, assuming $\sum_t \sum_i \sum_j Xo_{rj(t+1)} x_i > 0$, the problem is to maximize a fractional function. To convert the fractional model into a solvable equivalent form, an auxiliary variable Π is introduced. The important point is that Π is not assumed to be equal to the value of the objective function at this stage, but is defined as the lower bound of the value of the objective function. Therefore, the model is rewritten as Model (16):

$$\begin{aligned} &\max \Pi \\ &\text{s.t. } \Pi - \frac{\sum_i \sum_j \sum_t c_s\left(v_{it} VD_{ijt}\right)}{\sum_t \sum_i \sum_j Xo_{rj(t+1)} x_i} \leq 0, \\ &\frac{\sum_i \sum_t c_s\left(v_{it} VD_{ijt}\right)}{\sum_t \sum_r Xo_{rj(t+1)} x_{ij}} \leq 1 \quad \forall j, \\ &v_{it},\ Xo_{rj(t+1)} \geq 0. \end{aligned} \tag{16}$$

The reason for using the inequality in Model (16) is that, in a maximization problem, if Π is defined as the lower bound of the ratio and is maximized simultaneously, this inequality will necessarily become an equality at the optimal point; otherwise, it would be possible to increase Π without violating the constraints, which contradicts the optimality of the solution. Therefore, in the optimal solution, the following relationship holds:

$$\Pi^* = \frac{\left(\sum_i \sum_j \sum_t c_s\left(v_{it} VD_{ijt}\right)\right)^*}{\left(\sum_t \sum_i \sum_j Xo_{rj(t+1)} x_i\right)^*}. \tag{17}$$

The linear model equivalent to the fractional DEA model (15), obtained using Equation (17), is given in Model (18):

$$\begin{aligned} &\max \Pi \\ &\text{s.t. } \Pi \sum_t \sum_i Xo_{rj(t+1)} x_i - \sum_i \sum_t c_s\left(v_{it} VD_{ijt}\right) \leq 0, \\ &\sum_i \sum_t c_s\left(v_{it} VD_{ijt}\right) - \sum_t \sum_i Xo_{rj(t+1)} x_i \leq 0 \quad \forall j, \\ &v_{it},\ Xo_{rj(t+1)} \geq 0. \end{aligned} \tag{18}$$

Due to the absence of equality constraints in Model (18), inputs and outputs can be considered uncertain simultaneously. Finally, with the modifications made to the CCR form in the Bertsimas–Sim counterpart (14), constraints (8)–(11) can be rewritten as follows:

Corrected proofs

$$
\begin{aligned}
&\max\ \Pi\\
&\text{s.t. } \Pi\Big\{\sum_r\sum_j\sum_t Xo_{rj(t+1)}u_{rt}+\sum_j\sum_h\sum_t ym_{hj(t+1)}vm_{ht}\Big\}\\
&\qquad -c_s\Big(\sum_i\sum_j\sum_t v_{it}VD_{ijt}\Big)+c_b\Big(\sum_r\sum_j\sum_t Xo_{rj(t+1)}u_{rt}\Big)\le 0,\\
&\sum_r\sum_t u_{rt}B_{jt}+\sum_h\sum_t ym_{hj(t+1)}vm_{ht}-\sum_i\sum_t v_{it}\omega_{jt}A_{jt}\le 0\quad\forall j,\\
&\sum_r\sum_t Xo_{rj(t+1)}u_{rt}+\sum_h\sum_t vm_{ht}ym_{hj(t+1)}\\
&\qquad -\sum_i\sum_t VD_{ijt}v_{it}-\sum_t A_{jt}\omega_{jt}\le 0\quad\forall j,\\
&u_{rt},\ vm_{ht},\ \omega_{jt},\ v_{it}\ge\varepsilon.
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&\max\ \Pi\\
&\text{s.t. } c_s\Big(\sum_i\sum_j\sum_t v_{it}VD_{ijt}\Big)-c_b\Big(\sum_r\sum_j\sum_t Xo_{rj(t+1)}u_{rt}\Big)\\
&\qquad -\Pi\Big\{\sum_r\sum_j\sum_t Xo_{rj(t+1)}u_{rt}+\sum_j\sum_h\sum_t ym_{hj(t+1)}vm_{ht}\Big\}\\
&\qquad -\sum_j\sum_t\omega_{jt}-\Big(\sum_r\sum_t u_{rt}-\sum_h\sum_t vm_{ht}\Big)-z_i\Gamma\ge 0,\\
&\sum_r\sum_t Xo_{rj(t+1)}u_{rt}+\sum_h\sum_t vm_{ht}ym_{hj(t+1)}\\
&\qquad -\Pi\Big\{\sum_t\omega_{jt}A_{jt}+\sum_i\sum_t VD_{ijt}v_{it}\Big\}-\sum_r\sum_t\sum_j u_{rt}B_{jt}\\
&\qquad +\sum_h\sum_t ym_{hj(t+1)}vm_{ht}-\sum_t\omega_{jt}+\sum_r\sum_t u_{rt}+\sum_i\sum_t v_{it}\\
&\qquad +\sum_h\sum_t vm_{ht}+z_i'\Gamma\le 0\quad\forall j,\\
&\sum_r\sum_t Xo_{rj(t+1)}u_{rt}+\sum_t\sum_h vm_{ht}ym_{hj(t+1)}\\
&\qquad -\sum_i\sum_t VD_{ijt}v_{it}-\sum_t\sum_r A_{jt}\omega_{jt}+z_i''\Gamma\le 0\quad\forall j,\\
&z_i'+u_{rt}\ge\hat{\alpha}_r u_{rt}\quad\forall r,\qquad z_i'+v_{it}\ge\hat{\alpha}_i v_{it}\quad\forall i,\qquad z_i'+vm_{ht}\ge\hat{\alpha}_h vm_{ht}\quad\forall h,\\
&z_i''+u_{rt}\ge\hat{\alpha}_r u_{rt}\quad\forall r,\qquad z_i''+v_{it}\ge\hat{\alpha}_i v_{it}\quad\forall i,\qquad z_i''+vm_{ht}\ge\hat{\alpha}_h vm_{ht}\quad\forall h,\\
&u_{rt},\ vm_{ht},\ \omega_{jt},\ v_{it}\ge\varepsilon,\quad z_i,\ z_i',\ z_i''\ge 0.
\end{aligned}
\tag{20}
$$

In Model (19), inputs and outputs can be considered simultaneously uncertain due to the absence of equality constraints. Given the nonlinearity of Model (19), the main challenge in the robust model, which is more complex than the original model, is finding the optimal solutions.

Corrected proofs

The development of a robust NDEA model with uncertain inputs and outputs is presented in Model (20). In this model, Γ is the uncertainty budget, and z_i, z_i', z_i'' are the nonnegative dual variables associated with the budget constraints. The coefficients $\hat{\alpha}_r$, $\hat{\alpha}_i$, and $\hat{\alpha}_h$ denote the relative perturbation levels of the output, input, and intermediate-measure coefficients, respectively; in the computational experiments they are set equal to the robustness coefficient α reported in Table 3.

3.5 Lagrangian Relaxation Algorithm

Small and medium-sized instances of the model can generally be solved directly using optimization software. However, as the number of cryptocurrencies, indicators, and periods increases, the computational burden and memory requirements grow, and solving large instances with general-purpose solvers such as CPLEX becomes demanding. The Lagrangian relaxation method is widely used for solving complex optimization problems, particularly large-scale MILPs, by decomposing structured constrained models into tractable subproblems [29, 35, 65].

The main idea of the Lagrangian relaxation method is to relax complicating constraints by penalizing their violation in the objective function through Lagrange multipliers. As a result, solving the relaxed problem becomes computationally far more tractable than solving the original problem. Since the problem considered here is a maximization problem, for any fixed non-negative values of the Lagrange multipliers, the optimal value of the relaxed problem provides a valid upper bound (UB) for the original problem [29]. In other words, any solution to the relaxed problem gives an estimate of the optimal value of the original problem. Since complicating constraints are dualized and the feasible region is enlarged, solving the relaxed subproblems is much more straightforward. Besides, any feasible solution of the original problem provides a valid lower bound (LB). A heuristic procedure is typically employed to repair the relaxed solution into a primal feasible solution (LB). By minimizing the dual function over the Lagrange multipliers (e.g., via subgradient optimization), the tightest possible upper bound for the original problem is achieved [29].

3.5.1 Finding the Upper and Lower Bounds

We need to relax some complicating constraints to find the UB on the path to optimization. In this research, constraints (10) and (11) are the complicating constraints of the model. Since both constraints are stated for every cryptocurrency j, each is dualized with its own nonnegative multiplier, $\beta_{1j} \geq 0$ and $\beta_{2j} \geq 0$ ($j = 1, \ldots, n$); the dualized constraint violations are subtracted from the objective function. An LB is also found in each iteration of the Lagrangian relaxation method, whenever the relaxed solution can be repaired into a solution that is feasible for the

Corrected proofs

original problem. When the UB and LB are equal, the optimal solution is reached and the iterations are terminated. The general formula for Lagrangian relaxation for this problem is as follows:

$$\begin{aligned}\max\ & c_s\Big(\sum_i\sum_j\sum_t v_{it}VD_{ijt}\Big) - c_b\Big(\sum_r\sum_j\sum_t Xo_{rj(t+1)}u_{rt}\Big)\\ & -\sum_j\beta_{1j}\Big(\sum_r\sum_t u_{rt}B_{jt} + \sum_h\sum_t ym_{hj(t+1)}vm_{ht} - \sum_t \omega_{jt}A_{jt}\Big)\\ & -\sum_j\beta_{2j}\Big(\sum_r\sum_t Xo_{rj(t+1)}u_{rt} + \sum_h\sum_t vm_{ht}ym_{hj(t+1)} - \sum_i\sum_t VD_{ijt}v_{it} - \sum_t A_{jt}\omega_{jt}\Big)\end{aligned} \tag{21}$$

s.t.

$$\sum_r\sum_j\sum_t Xo_{rj(t+1)}u_{rt} + \sum_j\sum_h\sum_t ym_{hj(t+1)}vm_{ht} = 1 \tag{22}$$

3.5.2 Extensions to the Model

This section presents an extended linear programming model that considers reliability criteria during investment, as discussed below.

- The decision variable ω_{jt} is the (dimensionless) weight of cryptocurrency j in period t; the amount invested is $\omega_{jt}A_{jt}$. Since unlimited concentration in a single cryptocurrency does not exist in practice, ω_{jt} is limited to 10% (constraint (28)).
- A variable DT_{jt} is added to account for the supply shortfall of cryptocurrency j in period t, i.e., the amount by which the intended investment In_{jt} exceeds the available supply A_{jt}. A reliability constraint limits this shortfall to 5% of A_{jt} for each j and t.
- Constraint (29) explains situations where the supply of cryptocurrency is less than the investment amount. Finally, constraint (30) is added to limit this shortfall to a maximum of 5% of A_{jt}.

$$\max\ \Pi \tag{23}$$

$$\begin{aligned}\text{s.t. } & c_s\Big(\sum_i\sum_j\sum_t v_{it}VD_{ijt}\Big) - c_b\Big(\sum_r\sum_j\sum_t Xo_{rj(t+1)}u_{rt}\Big)\\ & -\Pi\Big\{\sum_r\sum_j\sum_t Xo_{rj(t+1)}u_{rt} + \sum_j\sum_h\sum_t ym_{hj(t+1)}vm_{ht}\Big\}\\ & -\sum_j\sum_t\omega_{jt} - \Big(\sum_r\sum_t u_{rt} - \sum_h\sum_t vm_{ht}\Big) - \sum_i z_i\Gamma \geq 0\end{aligned} \tag{24}$$

Corrected proofs

$$\sum_r \sum_t Xo_{rj(t+1)} u_{rt} + \sum_h \sum_t vm_{ht} ym_{hj(t+1)} - \Pi\Big\{\sum_t \omega_{jt} A_{jt} + \sum_i \sum_t VD_{ijt} v_{it}\Big\} - \sum_r \sum_t u_{rt} B_{jt} + \sum_h \sum_t ym_{hj(t+1)} vm_{ht} - \sum_t \omega_{jt} + \sum_r \sum_t u_{rt} + \sum_i \sum_t v_{it} + \sum_h \sum_t vm_{ht} + \sum_i z_i' \Gamma \leq 0 \quad \forall j \tag{25}$$

$$\sum_r \sum_t Xo_{rj(t+1)} u_{rt} + \sum_j \sum_h \sum_t vm_{ht} ym_{hj(t+1)} - \sum_i \sum_j \sum_t VD_{ijt} v_{it} - \sum_t \sum_j \sum_r A_{jt} \omega_{jt} + \sum_i z_i'' \Gamma \leq 0 \quad \forall j \tag{26}$$

$$\sum_r \sum_j \sum_t Xo_{rj(t+1)} u_{rt} + \sum_j \sum_h \sum_t ym_{hj(t+1)} vm_{ht} = 1 \tag{27}$$

$$\omega_{jt} \leq 0.1 \quad \forall j, t \tag{28}$$

$$DT_{jt} \geq In_{jt} - A_{jt} \quad \forall t, j \tag{29}$$

$$DT_{jt} \leq 0.05 A_{jt} \quad \forall t, j \tag{30}$$

$$z_i' + u_{rt} \geq \hat{\alpha}_r u_{rt} \ \ \forall r, \quad z_i' + v_{it} \geq \hat{\alpha}_i v_{it} \ \ \forall i, \quad z_i' + vm_{ht} \geq \hat{\alpha}_h vm_{ht} \ \ \forall h$$

$$z_i'' + u_{rt} \geq \hat{\alpha}_r u_{rt} \ \ \forall r, \quad z_i'' + v_{it} \geq \hat{\alpha}_i v_{it} \ \ \forall i, \quad z_i'' + vm_{ht} \geq \hat{\alpha}_h vm_{ht} \ \ \forall h$$

$$u_{rt},\ vm_{ht},\ \omega_{jt},\ v_{it} \geq \varepsilon, \quad DT_{jt} \geq 0, \quad z_i,\ z_i',\ z_i'' \geq 0 \tag{31}$$

4 Computational Results and Sensitivity Analysis

This research uses CoinMarketCap data to enhance the model's efficiency. These data are often employed to evaluate investment performance in digital cryptocurrencies using a network model, typically involving a two-stage process. The model variables are initially selected using financial criteria to evaluate cryptocurrencies from various perspectives, including risk, profitability, growth rate, and valuation. The data and variables for all investable cryptocurrencies are assessed based on the number of inputs and outputs. Therefore, 40 cryptocurrencies out of 613 digital currencies with the highest market transaction volume were selected for evaluation. The cryptocurrencies were chosen over six months. The developed model was coded and solved in GAMS software using the CPLEX solver. Assuming uncertainty in the input and output variables, the robust NDEA model with uncertain inputs and outputs was applied. Inputs for the first stage (cryptocurrency selection) include market value, price, 24-hour trading volume, and the ratio of trading volume to market value. On the other hand, the outputs of the first stage are 24-hour changes and the total market value of a digital asset. Assuming that all

Corrected proofs

outputs of the first stage are considered inputs for the second stage, in addition to the first stage outputs, transactions are also included as inputs for the second stage. The outputs of the second stage (investment amount) include the highest trading price, circulating supply, and maximum supply.

After data collection, the developed model considers some parameters uncertain due to market uncertainty to determine the efficiency of cryptocurrencies under different conditions. These uncertainties are addressed using robust optimization. Table 3 shows the results of implementing the robust NDEA model. According to the defined constraints, the numerical results indicate that with an increase in the robustness coefficient, i.e.,

$$\alpha = \{0.1,\ 0.25,\ 0.35,\ 0.45,\ 0.5,\ 0.65,\ 0.75,\ 0.85,\ 0.9\},$$

the objective function value worsens and the efficiency decreases. The total efficiency and the ranking of the ten highest-ranked cryptocurrencies are presented in Table 3. The cryptocurrencies are ranked by sorting them in descending order of their total efficiency scores EM_j obtained from the two-stage NDEA model (Equation (7)), so that rank 1 corresponds to the cryptocurrency with the highest overall efficiency. A DMU such as j is considered efficient if and only if the objective function of the model has an optimal solution equal to or close to 1. Otherwise, it is considered inefficient.

From a methodological perspective, the classical Mean-Variance framework relies exclusively on historical price returns and covariance structures, treating individual assets as single-stage black boxes. A well-documented limitation of this approach is the sensitivity of the optimal portfolio weights to estimation errors in the input parameters, particularly the expected returns [11]. Since cryptocurrency returns are highly volatile [41], such estimation errors are likely to be large in this market. In contrast, the two-stage robust NDEA model introduced here evaluates assets across both fundamental internal operations and financial risk–return dynamics, with robustness budgets controlling the level of conservatism. By simultaneously screening multidimensional performance dimensions and downside risk metrics under uncertainty, the proposed framework provides an alternative screening mechanism that helps avoid over-reliance on unstable covariance estimates.

Figure 3 shows the changes in the efficiency levels of cryptocurrencies, indicating that the efficiency ranking declines in the robust model for all cryptocurrencies. The highest efficiency is observed in the deterministic case. Additionally, evaluating efficiency under uncertain conditions shows that the ranking of different cryptocurrencies decreases as the range of fluctuations increases. The Lagrangian relaxation algorithm was then used to handle the complicating constraints in the problem. Figure 4 presents the variation of the objective function value with the Lagrange multiplier. The multipliers are the dual variables (shadow prices) of the dualized constraints: larger values penalize the violation of these constraints more heavily in the relaxed objective. They should not be interpreted as an investor's risk-aversion parameter.

Corrected proofs

Table 3: Results of the NDEA-robust model. For each value of α, the efficiency of the cryptocurrency selection stage (Sel.) and of the capital amount stage (Cap.) are reported. Total eff. is the overall efficiency of the two-stage NDEA model (Equation (7)), and Rank is obtained by sorting the cryptocurrencies in descending order of Total eff. (rank 1 = highest overall efficiency).

				$\alpha = 0.1$		$\alpha = 0.25$		$\alpha = 0.35$		$\alpha = 0.55$		$\alpha = 0.95$	
No.	Cryptocurrency	Rank	Total eff.	Sel.	Cap.	Sel.	Cap.	Sel.	Cap.	Sel.	Cap.	Sel.	Cap.
1	USDT	5	0.2456	0.1724	0.1945	0.1756	0.1987	0.1847	0.2032	0.1957	0.2224	0.2112	0.2415
2	BTC	1	0.8968	0.5847	0.4612	0.6454	0.5514	0.7929	0.6311	0.8781	0.7181	0.9141	0.8912
3	ETH	2	0.8621	0.5641	0.4513	0.6123	0.5321	0.7612	0.6203	0.8781	0.7181	0.9141	0.8912
4	USDC	4	0.4365	0.3841	0.3845	0.3978	0.3965	0.4056	0.4028	0.8621	0.7015	0.9014	0.8541
5	DOGE	3	0.8254	0.0361	0.7654	0.0375	0.7861	0.0396	0.7954	0.4169	0.4125	0.4236	0.4212
6	BNB	7	0.1725	0.1168	0.1265	0.1265	0.1356	0.1364	0.1456	0.1423	0.1547	0.1641	0.1612
7	SOL	6	0.2439	0.1856	0.1986	0.1987	0.2056	0.2056	0.2189	0.2154	0.2236	0.2245	0.2325
8	WLD	10	0.1396	0.2895	0.0932	0.2821	0.2954	0.2954	0.1028	0.3021	0.1145	0.3145	0.1241
9	XRP	8	0.1528	0.1078	0.1098	0.1135	0.1158	0.1158	0.1259	0.1256	0.1325	0.1456	0.1432
10	PEPE	9	0.1452	0.0956	0.0989	0.0987	0.1058	0.1058	0.1136	0.1145	0.1256	0.1254	0.1326

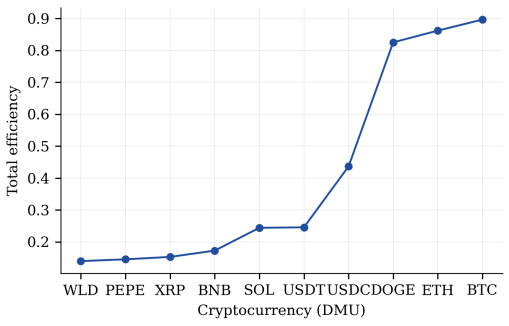


Figure 3: Changes in the efficiency level of cryptocurrencies.

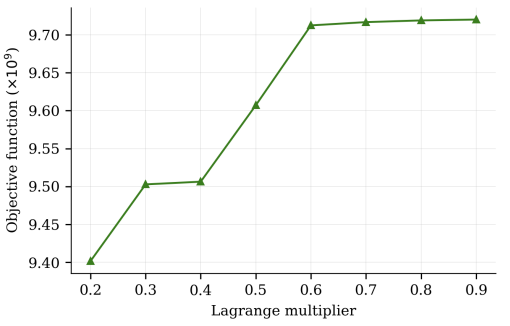


Figure 4: The effect of the Lagrange multiplier on the objective function.

Corrected proofs

The Lagrangian relaxation method addresses the complicating constraints of the model and determines the LB and UB. Figure 5 shows how the UB and LB of the model converge. As the gap between the UB and LB decreases, the solution gets closer to the optimal one.

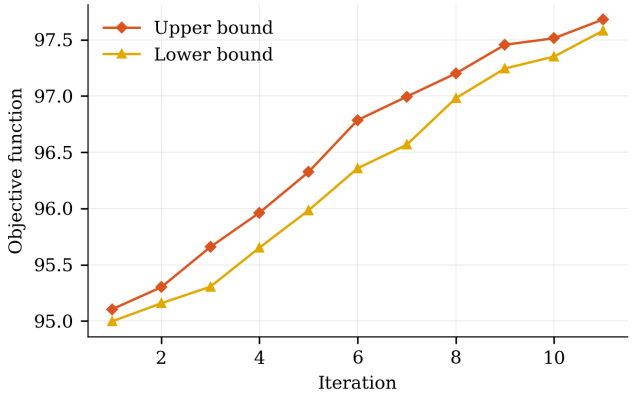


Figure 5: Convergence of the upper and lower bounds in the Lagrangian relaxation algorithm.

Table 4 presents the computation time results for the deterministic method (GAMS) and the Lagrangian relaxation method. The LB and UB are shown in the second and third columns of the table, respectively. The optimality gap is the relative difference between the UB and the LB, calculated as $\text{Gap} = (\text{UB} - \text{LB})/\text{UB}$. Since the problem is a maximization problem, the UB is the value of the Lagrangian dual (relaxed) problem, and the LB is the objective value of the best feasible solution of the original problem obtained by the heuristic procedure. Table 4 reports the bounds, gaps, and running times of both approaches. GAMS with CPLEX returns a solution for the first three instances, in shorter running times than the Lagrangian relaxation algorithm, whereas no GAMS solution is reported for instances 4–10. The Lagrangian relaxation algorithm returns bounds for all ten instances, and its running time increases with the problem size.

Relaxing the complicating constraints simplifies the subproblem solved in each iteration. The algorithm stops when the gap between the UB and the LB falls below a prescribed tolerance; if the two bounds coincide, the solution is optimal.

In this research, the Lagrange multiplier method was applied with values from two different periods to obtain the best solution with minimal error. The periods are denoted by p ($p = 5$ for the first period and $p = 3$ for the second period), and the number of boundary points is denoted by m. Boundary points are the optimal and efficient points lying on the efficiency frontier (the production possibility frontier), i.e., the cryptocurrencies that attain an efficiency score of one with zero slacks. The Lagrange multipliers were updated iteratively by the subgradient method [6], in which the step sizes are adjusted according to the magnitude of the violation of the relaxed constraints (uncertainty and capacity constraints), so that the error decreases over successive iterations. The percentage error of the upper bound (UB) is computed as the optimality gap

Corrected Proofs

Table 4: Comparison of the deterministic method (GAMS) and the Lagrangian relaxation algorithm.

	Lagrangian relaxation algorithm				GAMS (network data envelopment analysis)		
Problem number	LB	UB	Gap (%)	Running time	Objective value	Gap	Running time
1	94.105	95.301	0.01	0.416	94.111	–	0.027
2	95.301	95.658	0.23	0.451	95.154	–	0.035
3	95.305	95.325	0.95	0.575	96.318	–	0.201
4	95.652	96.325	0.22	0.674	–	–	–
5	96.519	96.786	6.3	0.777	–	–	–
6	96.651	96.995	8	0.783	–	–	–
7	96.981	97.201	12.6	0.884	–	–	–
8	97.245	97.456	21.9	0.890	–	–	–
9	97.351	97.514	16.3	1.205	–	–	–
10	97.681	97.581	10	1.215	–	–	–

used in Table 4, i.e., $(\text{UB} - \text{LB})/\text{UB}$ expressed as a percentage. For the UBs, the errors lie between 1% and 5% at $p = 5$ and $m = 5$ (Figure 6), with an average error of 0.18. For the UBs, the errors lie between 5% and 9% at $p = 3$ and $m = 3$, with an average error of 1; according to Figure 7, this is the worst solution for the objective function compared with period 1. The results indicate that the more boundary points there are, the lower the errors.

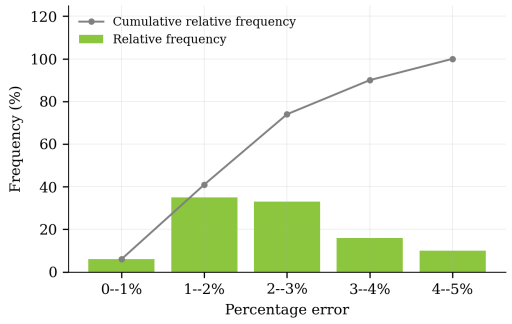


Figure 6: Analysis of percentage errors at upper bounds in the first setting ($p = 5, m = 5$).

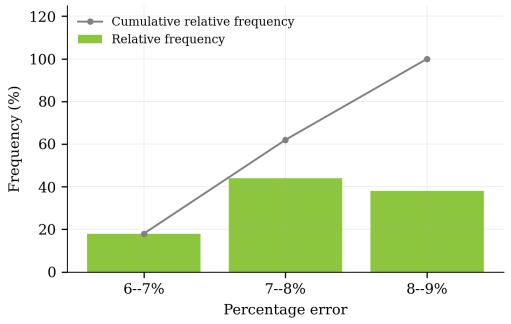


Figure 7: Analysis of percentage errors at upper bounds in the second setting ($p = 3, m = 3$).

5 Conclusion

This research developed a combined optimization model using both Data Envelopment Analysis (DEA) and robust optimization to evaluate risk with uncertain inputs and outputs. This study

Corrected proofs

provides a framework for selecting an optimal portfolio and assessing the model's reliability with an open structure at each stage. In each stage, the parameters for the level of investment and cryptocurrency selection are considered under uncertain robust planning. By setting conservatism, the robust optimization approach controls uncertainty in input and output data simultaneously without identifying the data distribution. The results show that as disturbances increase, the overall efficiency in the first phase decreases; this decrease in efficiency scores is visible for each DMU. Due to strict constraints in the developed models and reduced efficiency levels, fewer units are on the efficiency frontier, eliminating the need for super-efficiency models for re-evaluating efficient units. Additionally, the ability to rank and differentiate the proposed cryptocurrencies is evident in calculating the efficiency of decision-making units. Furthermore, the efficiency level of the robust models varies according to the level of conservatism and the degree of parameter disturbances, acting effectively in evaluating efficiency.

In this study, the investment process in the cryptocurrency market is modeled as a multi-stage system. The proposed network structure includes stages that represent, respectively, the assessment of risk and market characteristics, the transformation of these characteristics into intermediate performance indicators, and finally the realization of investment returns. In this framework, intermediate variables serve as information transmitters between stages of the investment process and enable modeling of the gradual transformation of risk and market conditions into the asset's final performance. This network structure is particularly well-suited to the cryptocurrency market, which is characterized by high volatility, continuous trading, and dynamic capital flows across different time periods. In such a situation, evaluating performance based solely on a direct relationship between inputs and outputs may not provide a complete picture of the investment process. Using the NDEA framework allows for the separation of different stages of value creation in the investment process, thereby reflecting the impact of market fluctuations and changes in capital flows in a step-by-step manner.

At the same time, the proposed framework is not conceptually limited to the cryptocurrency market. Since the NDEA model is primarily designed to analyze multi-stage systems, the proposed structure can be extended to other financial markets by appropriately defining input, intermediate, and output variables. For example, in the stock market, similar stages such as risk assessment and liquidity characteristics, the formation of intermediate performance indicators, and the realization of the final investment return can be modeled in the form of a multi-stage network. Therefore, while the design of variables in this study has been carried out with respect to the characteristics of the cryptocurrency market, the methodological framework of the model can also be adapted and applied to other financial markets with different structures.

Despite its theoretical and practical contributions, this study has certain limitations. First, the model was empirically tested using only the 40 cryptocurrencies with the highest trading volume (based on CoinMarketCap data); this likely limits the generalizability of the results to emerging cryptocurrencies, low-market-cap cryptocurrencies, or cryptocurrencies with low

Corrected proofs

liquidity. Second, the robust network data envelopment analysis model accounts for data uncertainty via interval-based representation but does not explicitly incorporate inter-asset correlations or the nonlinear dynamics of market volatility. Third, the selection of input, intermediate, and output indicators (such as NVT, RSI, OBV, and growth metrics) was tailored to the structure of the cryptocurrency market; consequently, the model's applicability to traditional markets (e.g., stock exchanges) requires independent empirical evaluation. Finally, the Lagrangian relaxation algorithm yields near-optimal solutions with a reported optimality gap rather than guaranteeing a definitive global optimum.

The numerical results indicate that the efficiency scores of the robust model decrease as the conservatism level increases, which is the price paid for protecting the evaluation against perturbations of the data. The Lagrangian relaxation algorithm was applied to obtain upper and lower bounds on the optimal value of the model, and the reported gaps measure the quality of the solutions obtained; it provides near-optimal solutions rather than a guarantee of global optimality. In the numerical comparison of Table 4, direct solution with GAMS/CPLEX returned solutions only for the first three instances, whereas the Lagrangian relaxation algorithm returned bounds for all instances. The following directions are proposed for future research.

- Designing multi-period dynamic models enables the analysis of investment accumulation effects across various market styles.
- Combining Lagrangian relaxation with exact decomposition methods (such as Benders decomposition) or hybrid metaheuristics can tighten dual bounds and improve solution quality.
- Future research should employ ellipsoidal uncertainty sets or copula-based structures to account for asset interdependencies. Additionally, integrating robust optimization with time-varying parameter vector autoregression (TVP-VAR) models can align the conservatism parameter with structural market changes.
- A robust NDEA model based on variable returns to scale (VRS) facilitates a more accurate assessment of assets with vastly different scales.

Declarations

Availability of Supporting Data

All data generated and analyzed during this research is published in this article. The data is extracted from the site https://coinmarketcap.com

Funding

Corrected proofs

The authors received no funding or financial support for this research.

Conflict of Interest

The authors declare that they have no conflict of interest.

Authors' Contributions

Alireza Hamidieh: Supervision; Project administration; Writing – review & editing; Guidance. **Bahareh Akhgari Rik**: Methodology; Modeling; Formal analysis; Writing – original draft preparation.

Artificial Intelligence Statement

Both authors used generative AI (ChatGPT) solely for language refinement, grammar correction, and improving readability. No AI tool was used for mathematical derivations, scientific analysis, data generation, interpretation of results, or scientific conclusions.

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Corrected proofs