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Numerical Solution of the Controlled Harmonic Oscillator by Homotopy Perturbation Method

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Abstract. The controlled harmonic oscillator with retarded damping, is an important class of optimal control problems which has an important role in oscillating phenomena in nonlinear engineering systems. In this paper, to solve this problem, we presented an analytical method. This approach is based on the homotopy perturbation method. The solution procedure becomes easier, simpler and more straightforward. In order to use the proposed method, a control design algorithm with low computational complexity is presented. Through the finite iterations of the proposed algorithm, a suboptimal control law is obtained for the problems. Finally, the obtained results have been compared with the exact solution of the controlled harmonic oscillator and variational iteration method, so that the high accuracy of the results is clear.

Keywords. Suboptimal control, Harmonic oscillator, Damping, Homotopy perturbation method.

MSC. 34H05; 49J15.

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1 Introduction

The controlled harmonic oscillator, which is known to describe many important oscillating phenomena in nonlinear engineering systems [24], received considerable attention in the past decade.
The classical Duffing's equation was first introduced to study electronics and was published by Stokes in [26]. It is the simplest oscillator displaying catastrophic jumps of amplitude and phase when the frequency of the forcing term is taken as a gradually changing parameter. The Duffing equation has wide applications in signal processing [29], the propagation of extremely short electromagnetic pulses in a nonlinear medium [15, 16], brain modeling [30], fuzzy modeling and the adaptive control of uncertain chaotic systems [6, 22].

Vlassenbroeck and Van Dooren [3] introduced a direct method for the controlled Duffing oscillator. "In the study of Vlassenbroeck et al. the state and control variables, the system dynamics, and the boundary conditions have been expanded in Chebyshev series of order with unknown coefficient. A pseudospectral collocation method for solving the nonlinear controlled Duffing oscillator was presented in [21]. This approach is based on the idea of relating Legendre-Gauss-Lobatto collocation and points to the structure of orthogonal polynomials. Elnagar and Khamayseh [5] presented an alternative computational method for solving the controlled Duffing oscillator. Their approach drew upon the power of well-developed nonlinear programming techniques and computer codes to determine the optimal solutions of nonlinear systems. El-Kady and Elbarbary [4] used Chebyshev polynomials for solving controlled Duffing oscillator. In [4], the control and state variables are approximated by Chebyshev series of different orders. Marzban and Razzaghi [17] introduced an alternative computational method for solving the controlled Duffing oscillator. Rad et al. [20] presented a new numerical method which is applied to investigate the nonlinear controlled Duffing oscillator. This method is based on the radial basis functions (RBFs) to approximate the solution of the optimal control problem by using the collocation method.

Recently, scientists and engineers have become highly eager to find the analytic solution to nonlinear problems; for this purpose, many new techniques and methods have been developed. Traditional perturbation methods have their own limitations for example, the presence of a very large or very small parameter inside the problem is essential, so that the solution of the problem may be expressed as a series expansion in terms of that small parameter. Choosing the small parameter is not an easy task and requires special skills. A proper and good choice of a small parameter will make results more accurate, while, on the other, hand a wrong choice may lead to inaccurate results. The homotopy perturbation method (HPM), first proposed by He [8] has successfully been applied to solve many types of linear and nonlinear functional equations. This method, which is a combination of homotopy in topology and classic perturbation techniques, provides us with a convenient way to obtain analytic or approximate solutions for a wide variety of problems arising in different fields (see e.g.,[1, 12, 23]). The other method variational iteration method (VIM) techniques in various types of problems, and many new methods have been introduced into the literature. This method was introduced by the Chinese mathematician He [9] and [10] first, by modifying the general Lagrange multiplier method. The main idea in the VIM is to construct an iterative sequence of functions, to be converted to an exact solution. Since the method works without discretization, linearization, transformation, or perturbation of the problem, it is not affected by round of error (see e.g., [18, 19, 25, 27, 28]).

In this paper, our aim is to develop the HPM and VIM methods for solving the controlled harmonic oscillator. We present a comparative study with these two methods. The study outlines the significant features of the HPM and VIM methods. By applying necessary optimality conditions, we obtain iterative formulas for the HPM and VIM. By using the finite-step iteration of algorithm, we can obtain a suboptimal control law. The convergence of the HPM is studied and to illustrate the effectiveness of these methods, some test problems are investigated. In a word, the HPM and VIM show that the techniques are reliable, powerful and promising methods for controlled harmonic oscillator with retarded damping.

The structure of this paper is arranged as follows: Section 2 is devoted to Pontryagin's maximum principle used for solving the controlled harmonic oscillator. Section 3 is dedicated to the proposed design approach for solving a close-loop optimal control problem based on the HPM and convergence of the method is demonstrated. Section 4 is devoted to the suboptimal control strategy and algorithm for the proposed method. In Section 5 the numerical examples are simulated to show the reasonableness of our theory and demonstrate the performance of our network. Finally, we end this paper with conclusions in Section 6.

2 The Controlled Linear Oscillator

Consider the optimum control of a linear oscillator governed by the differential equation:

$$\begin{cases} \ddot{x}(t) + a\dot{x}(t) + b\dot{x}(t-\tau) + cx(t) = u(t), \quad t_0 \leqslant t \leqslant t_f, \\ x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$ are respectively the state vector and the control vector, $x_0 \in \mathbb{R}^n$ is the initial state vector, c is the stiffness parameter, and a, b are the viscous damping coefficients. Note that an artificially produced damping term $b\dot{x}(t-\tau)$ is added to help control or stabilize a system with insufficient natural damping $a\dot{x}(t)$.
The objective is to find optimal control law $u^*(t)$ which minimizes the following quadratic cost functional:

$$J = \frac{1}{2}x^T(t_f)Q_f x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \Big(x^T(t)Q(t)x(t) + u^T(t)R(t)u(t) \Big) dt, \tag{2}$$

where, matrix $Q_f \in \mathbb{R}^{n\times n}$ is symmetric positive semi-definite, matrix $Q(t) \in \mathbb{R}^{n\times n}$ is symmetric positive semi-definite and piecewise continuous for $t_0 \leq t \leq t_f$, and matrix $R(t) \in \mathbb{R}^{m\times m}$ is symmetric positive definite with appropriate dimensions.

Equation (1) is equivalent to the dynamic state equations:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -cx_1(t) - ax_2(t) - bx_2(t-\tau) + u(t) \\ x_1(t_0) = x_0, \quad x_2(t_0) = \dot{x}_0, \end{cases} \tag{3}$$

the problem is to find control vector $u(t)$ which minimizes equation (2) subject to equation (3).

The Hamiltonian function for the problem is

$$\mathcal{H}(x,u,\lambda,t)=\frac{1}{2}x^T(t)Q(t)x(t)+\frac{1}{2}u^T(t)R(t)u(t)+\lambda^T(t)[Ax(t)+A_1x(t-\tau)+Bu(t)],$$

where

$$A=\begin{pmatrix}0 & 1\\ -c & -a\end{pmatrix},\quad A_1=\begin{pmatrix}0 & 0\\ 0 & -b\end{pmatrix},\quad B=\begin{pmatrix}0\\ 1\end{pmatrix}.$$

According to the Pontryagin's maximum principle of optimal control problems with time-delay (1), the necessary conditions of optimality can be written as follows [13]:

$$\begin{cases}\dot{x}(t)=Ax(t)+A_1x(t-\tau)-S\lambda(t) & t_0\leqslant t\leqslant t_f\\ \dot{\lambda}(t)=\begin{cases}-Qx(t)-A^T\lambda(t)-A_1^T\lambda(t+\tau) & t_0\leqslant t<t_f-\tau\\ -Qx(t)-A^T\lambda(t) & t_f-\tau\leqslant t\leqslant t_f\end{cases}\\ x_0(t)=x(t_0) & t_0-\tau\leq t\leqslant t_0\\ \lambda(t_f)=Q_fx(t_f),\end{cases}\tag{4}$$

where $S=BR^{-1}B^T$, $x(t-\tau)$ is the time-delay term and $\lambda(t+\tau)$ is the time-advance term; furthermore, $\lambda(t)\in PC^1([t_0,t_f],\mathbb{R}^n)$ is the co-state vector. In addition, the optimal control law is given by:

$$u^*(t)=-R^{-1}B^T\lambda(t)\qquad t_0\leqslant t\leqslant t_f.\tag{5}$$

Note that, equation (4) forms a linear two-point boundary value problem (TPBVP) with time-varying coefficient involving both delay and advance terms. The exact solution to this problem is, in general, extremely difficult, if not impossible. In recent decades, some new numerical and analytic approximate methods have been proposed for solving such a difficult problem in the context of delay ordinary differential equations. To overcome this difficulty, an iterative approach, based on the HPM, will be introduced in the next section.

3 He's HPM and Optimal Control Design Strategy

The homotopy perturbation method is a combination of the classical perturbation technique and homotopy concept as used in topology. To explain the basic of the homotopy perturbation method for solving nonlinear differential equations, the following nonlinear differential equation is considered:

$$A(u)-f(r)=0\quad r\in\Omega,\tag{6}$$

subject to the boundary condition

$$B(u,\frac{\partial u}{\partial n})=0\quad r\in\Gamma,\tag{7}$$

where A is an integral differential operator, B is a boundary operator, $f(r)$ is a known analytical function, Γ is the boundary of domain Ω, and $\frac{\partial}{\partial n}$ denotes differentiation along the normal drawn outwards from Ω.

The operator A can, generally speaking, be divided into two parts, a linear part $\mathcal{L}$ and a nonlinear part $\mathcal{N}$. Therefore, (6) can be written as follows:

$$\mathcal{L}(u) + \mathcal{N}(u) - f(r) = 0. \tag{8}$$

By the homotopy technique, He constructed a homotopy $v(r,p) : \Omega \times [0,1]; r \in \Omega, p \in [0,1]$ which satisfies:

$$H(v,p) = (1-p)[\mathcal{L}(v) - \mathcal{L}(u_0)] + p[A(v) - f(r)] = 0, \tag{9}$$

which is equivalent to

$$H(v,p) = \mathcal{L}(v) - \mathcal{L}u_0 + p\mathcal{L}u_0 + p[\mathcal{N}(v) - f(r)] = 0, \tag{10}$$

where $p \in [0,1]$ is an embedding parameter, and u_0 is an initial guess approximation of (6), which satisfies the boundary conditions. It follows from (9) that

$$H(v,0) = \mathcal{L}(v) - \mathcal{L}(u_0) = 0,$$
$$H(v,1) = A(v) - f(r) = 0.$$

Thus, the process of changing p from zero to unity is just like that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and $\mathcal{L}(v) - \mathcal{L}(u_0)$ and $A(v) - f(r)$ are called homotopic. Herein, the embedding parameter p as a small parameter is used and is assumed that the solution of equation (9) or (10) is as a power series in ρ:

$$v = \sum_{k=0}^{\infty} p^k v_k = v_0 + pv_1 + p^2 v_2 + \cdots . \tag{11}$$

Setting $p \to 1$, the approximate solution of equation (6) is obtained:

$$u = \lim_{p\to 1} v = \sum_{k=0}^{\infty} v_k. \tag{12}$$

The convergence of series (12) has been proved in [8].

Theorem 1. Suppose that $\mathcal{N}(v)$ is a nonlinear function, and $v = \sum_{k=0}^{\infty} p^k v_k$, then we have

$$\frac{\partial^n}{\partial p^n}\mathcal{N}(v)_{p=0} = \frac{\partial^n}{\partial p^n}\mathcal{N}(\sum_{k=0}^{\infty} p^k v_k)_{p=0} = \frac{\partial^n}{\partial p^n}\mathcal{N}(\sum_{k=0}^{n} p^k v_k)_{p=0}.$$

Proof. Since

$$v = \sum_{k=0}^{\infty} p^k v_k = \sum_{k=0}^{n} p^k v_k + \sum_{k=n+1}^{\infty} p^k v_k,$$

we have the following result

$$\frac{\partial^n}{\partial p^n}\mathcal{N}(v)_{p=0} = \frac{\partial^n}{\partial p^n}\mathcal{N}(\sum_{k=0}^{\infty} p^k v_k)_{p=0}$$
$$= \frac{\partial^n}{\partial p^n}\mathcal{N}\Big(\sum_{k=0}^{n} p^k v_k + \sum_{k=n+1}^{\infty} p^k v_k\Big)_{p=0} = \frac{\partial^n}{\partial p^n}\mathcal{N}(\sum_{k=0}^{\infty} p^k v_k).$$

This ends the proof. □

Theorem 2. The approximate solution of (6) obtained by the HPM can be expressed in He's polynomials:

$$u(r) = f(r) + H_0(v_0) + H_1(v_0, v_1) + \cdots + H_n(v_0, v_1, \cdots, v_n),$$

where He's polynomials are defined as follows:

$$H_j(v_0, v_1, \cdots, v_j) = -\mathcal{L}^{-1}\Big(\frac{1}{j!}\frac{\partial^j}{\partial p^j}\mathcal{N}(\sum_{k=0}^{j} p^k v_k)\Big), \quad j = 0, 1, 2, \cdots n.$$

Proof. For more details, See Ghorbani [7]. □

In the following, to perform this methodology for solving nonlinear TPBVP (4), we construct $\tilde{x}(t,p) : [t_0, t_f] \times [0,1] \to \mathcal{R}^n$ and $\tilde{\lambda}(t,p) : [t_0, t_f] \times [0,1] \to \mathcal{R}^n$, which satisfy:

$$\begin{aligned} H(\tilde{x}(t,p), \tilde{\lambda}(t,p), p) &= \begin{pmatrix} H_1(\tilde{x}(t,p), \tilde{\lambda}(t,p), p) \\ H_2(\tilde{x}(t,p), \tilde{\lambda}(t,p), p) \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{L}_1(\tilde{x}(t,p), \tilde{\lambda}(t,p)) - \mathcal{L}_1(x_0(t), \lambda_0(t)) + p\mathcal{L}_1(x_0(t), \lambda_0(t)) + p\mathcal{N}_1(\tilde{x}(t,p), \tilde{\lambda}(t,p)) \\ \mathcal{L}_2(\tilde{x}(t,p), \tilde{\lambda}(t,p)) - \mathcal{L}_2(x_0(t), \lambda_0(t)) + p\mathcal{L}_2(x_0(t), \lambda_0(t)) + p\mathcal{N}_2(\tilde{x}(t,p), \tilde{\lambda}(t,p)) \end{pmatrix} \\ &= 0, \end{aligned} \tag{13}$$

where $p \in [0,1]$ is the homotopy parameter and $x_0(t), \lambda_0(t)$ is an initial approximation for the solution of nonlinear TPBVP (4). In addition, the linear and nonlinear operators in (13), i.e., $\mathcal{L}_1, \mathcal{L}_2, \mathcal{N}_1$ and $\mathcal{N}_2$ are defined as follows:

$$\begin{cases} \mathcal{L}_1(\tilde{x}(t,p), \tilde{\lambda}(t,p)) = \frac{\partial \tilde{x}(t,p)}{\partial t} - Ax(t,p) + S\lambda(t,p), \\ \mathcal{L}_2(\tilde{x}(t,p), \tilde{\lambda}(t,p)) = \frac{\partial \tilde{\lambda}(t,p)}{\partial t} + Q\tilde{x}(t,p) + A^T\tilde{\lambda}(t,p), \\ \mathcal{N}_1(\tilde{x}(t,p), \tilde{\lambda}(t,p)) = -A_1\tilde{x}(t-\tau, p), \\ \mathcal{N}_2(\tilde{x}(t,p), \tilde{\lambda}(t,p)) = \begin{cases} A_1\tilde{\lambda}(t+\tau, p), & t_0 \le t < t_f - \tau, \\ 0, & t_f - \tau \le t \le t_f. \end{cases} \end{cases} \tag{14}$$

In addition, the initial approximations, $x_0(t)$ and $\lambda_0(t)$, are chosen to be the solution of the following linear time-invariant TPBVP:

$$\begin{cases} \mathcal{L}_1(x_0(t), \lambda_0(t)) = 0, \\ \mathcal{L}_2(x_0(t), \lambda_0(t)) = 0, \\ x_0(t) = x(t_0), \quad \lambda_0(t) = Q_f x_0(t_f). \end{cases} \tag{15}$$

From (13) we obtain

$$H(\tilde{x}(t,0),\tilde{\lambda}(t,0),0)=\begin{pmatrix}\mathcal{L}_1(\tilde{x}(t,0),\tilde{\lambda}(t,0))-\mathcal{L}_1(x_0(t),\lambda_0(t))\\ \mathcal{L}_2(\tilde{x}(t,0),\tilde{\lambda}(t,0))-\mathcal{L}_2(x_0(t),\lambda_0(t))\end{pmatrix}=0, \tag{16}$$

$$H(\tilde{x}(t,1),\tilde{\lambda}(t,1),1)=\begin{pmatrix}\mathcal{L}_1(\tilde{x}(t,1),\tilde{\lambda}(t,1))+\mathcal{N}_1(\tilde{x}(t,1),\tilde{\lambda}(t,1))\\ \mathcal{L}_2(\tilde{x}(t,1),\tilde{\lambda}(t,1))+\mathcal{N}_2(\tilde{x}(t,1),\tilde{\lambda}(t,1))\end{pmatrix}=0. \tag{17}$$

Equations (16) and (17) implicitly state that, when p increases from zero to one, the trivial problem (16) is continuously deformed to problem (17).

According to the HPM, the homotopy parameter p is used to expand solutions $\tilde{x}(t,p)$ and $\tilde{\lambda}(t,p)$ in the form:

$$\begin{cases}\tilde{x}(t,p)=\tilde{x}_0(t)+p\tilde{x}_1(t)+p^2\tilde{x}_2(t)+\cdots+=\sum_{i=0}^{\infty}p^i\tilde{x}_i(t),\\ \tilde{\lambda}(t,p)=\tilde{\lambda}_0(t)+p\tilde{\lambda}_1(t)+p^2\tilde{\lambda}_2(t)+\cdots+=\sum_{i=0}^{\infty}p^i\tilde{\lambda}_i(t),\end{cases} \tag{18}$$

where $\tilde{x}_i(t)=\frac{1}{i!}\frac{\partial\tilde{x}(t,p)}{\partial p^i}|_{p=0}$ and $\tilde{\lambda}_i(t)=\frac{1}{i!}\frac{\partial\tilde{\lambda}(t,p)}{\partial p^i}|_{p=0}$.

Setting $p=1$ in (18), we get

$$\begin{cases}x(t)=\lim_{p\to 1}\tilde{x}(t,p)=\tilde{x}_0(t)+\tilde{x}_1(t)+\tilde{x}_2(t)+\cdots+=\sum_{i=0}^{\infty}\tilde{x}_i(t),\\ \lambda(t)=\lim_{p\to 1}\tilde{\lambda}(t,p)=\tilde{\lambda}_0(t)+\tilde{\lambda}_1(t)+\tilde{\lambda}_2(t)+\cdots+=\sum_{i=0}^{\infty}\tilde{\lambda}_i(t).\end{cases} \tag{19}$$

By substituting $\tilde{x}(t,p)$ and $\tilde{\lambda}(t,p)$ from (18) into (13), rearranging with respect to the powers of p, and then equating the coefficients of the same powers of p, $\tilde{x}^{(i)}(t)$ and $\tilde{\lambda}^{(i)}(t)$ for $i\geq 0$ can be easily obtained in the i^{th} step only by solving the nonhomogeneous linear time-invariant TPBVP. In addition, in each step, nonhomogeneous terms are calculated using the information obtained from previous steps. Hence, solving the presented sequence is a recursive process.

Finally, according to previous discussions, the following theorem can be stated:

Theorem 3. Consider system (1) with quadratic cost functioal (2). Then, the optimal trajectory and the optimal control law for $t\in[t_0,t_f]$ are determined as follows:

$$\begin{cases}x^*(t)=\sum_{i=0}^{\infty}\tilde{x}_i(t),\\ u^*(t)=-R^{-1}B^T\sum_{i=0}^{\infty}\tilde{\lambda}_i(t),\end{cases} \tag{20}$$

where $\tilde{x}_i(t)$ and $\tilde{\lambda}_i(t)$ for $i\geq 0$ are obtained only by solving recursively the presented sequence of TPBVP in (14).

Proof. It is straightforward by using Theorem 2 and equations (13)-(19). □

4 Suboptimal Control Design Strategy

In this section we explain how to evaluate the precision of the suboptimal manner. We may find the suboptimal control law in practical applications by replacing infinite with a finite positive integer N in (20). Thus, the N^{th} order suboptimal trajectory-control pair is obtained as follows:

$$\begin{cases} x_N(t) = \sum_{i=0}^{N} \tilde{x}_i(t), \\ u_N(t) = -R^{-1}B^T \sum_{i=0}^{N} \tilde{\lambda}_i(t). \end{cases} \tag{21}$$

The integer N^{th} in (21) is generally determined according to a concrete control precision. For example, every time $\tilde{x}_i(t)$ and $\tilde{\lambda}_i(t)$ are obtained from the presented linear TPBVP sequences, we let $N = i$ and calculate $x_N(t)$ and $u_N(t)$ from (21). Then, the following quadratic performance index can be calculated:

$$J_N = \frac{1}{2}x_N^T(t_f)Q_f x_N(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left(x_N^T(t)Q(t)x_N(t) + u_N^T(t)R(t)u_N(t) \right) dt, \tag{22}$$

where $u_N(t)$ has been obtained from (21) and $x_N(t)$ is the corresponding state trajectory obtained from applying $u_N(t)$ in (1).

For the accuracy analysis, the following criterion is considered. The suboptimal control law has desirable accuracy; if for given positive constants $\epsilon > 0$, the following condition holds jointly:

$$\left| \frac{J_N - J_{N-1}}{J_N} \right| < \epsilon, \tag{23}$$

If the tolerance error bound is chosen small enough, the N^{th} order suboptimal control law will be very close to the optimal control law $u^*(t)$; thus, the value of quadratic performance index in (22) and its optimal value J^* will be almost identical, according to Theorem 2, and the boundary state conditions will be satisfied tightly.

Now, in order to maintain the accuracy of solutions, we present an algorithm of the proposed method with low computational complexity.

Algorithm: Suboptimal control law of system (1):

1. Obtain $x_0(t)$ and $\lambda_0(t)$ from (13). Set $\tilde{x}_0(t) = x_0(t)$ and $\tilde{\lambda}_0(t) = \lambda_0(t)$. Let $i = 1$.
2. Calculate the ith order terms $\tilde{x}_i(t)$ and $\tilde{\lambda}_i(t)$ from (13).
3. Let $i = N$ and calculate $x_N(t)$ and $u_N(t)$ from (21).
4. Calculate J_N according to (22). If $\left| \frac{J_N - J_{N-1}}{J_N} \right| < \epsilon$, then stop and output $u_N(t)$, go to step 5; else, replace i by $i + 1$ and go to step 2.
5. Stop the algorithm; $x_N(t)$ and $u_N(t)$ are accurate enough.

5 Illustrative Examples

The following various examples are given to illustrate the simplicity and efficiency of the proposed method. The codes are developed using symbolic computation software MATLAB and the calculations are implemented on a machine with Intel core 2 Due processor 2.50 Ghz and 4 GB RAM.

Example 1. Consider the optimal control problem of a harmonic oscillator with retarded damping as in [2]

$$J = 5x_1^2(t) + \frac{1}{2}\int_0^2 u^2 dt, \tag{24}$$

subject to:

$$\begin{cases} \ddot{x}(t) + \dot{x}(t-1) + x(t) = u(t), \\ x(0) = 10, \quad \dot{x}(0) = 0, \end{cases} \tag{25}$$

where $0 \le t \le 2$. Equation (25) is equivalent to

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -x_1(t) - x_2(t-1) + u(t), \\ x_1(0) = 10, \quad x_2(0) = 0. \end{cases} \tag{26}$$

The problem is to find the control vector $u(t)$ which minimizes equation (24) subject to equation (26). The exact solution for this problem is given by [2]:

$$u^*(t) = \begin{cases} \delta\sin(2-t) + (\frac{\delta}{2})(1-t)\sin(t-1) & 0 \leqslant t < 1, \\ \delta\sin(2-t) \qquad 1 \leqslant t \leqslant 2. \end{cases}$$

where $\delta = 2.5599$, and the optimal cost functional is $J^* = 3.3991$. Implementing the algorithm described in section 4 with the tolerance error bounds $\epsilon = 3 \times 10^{-5}$, the desired suboptimal control is obtained for $N = 7$ iterations. From Table 1, it is observed that, $\left|\frac{J_7 - J_6}{J_7}\right| = 2.941 \times 10^{-5} < \epsilon$, and a minimum value of $J_7 = 3.3991$ is obtained. Note that dash in tables 1 and 3 indicates that the error's value can not be calculated at the first step. The error of proposed method can be calculated for the later steps. In Table 2, a comparison is made between the value of J obtained by the present method with $N = 7$, together with the value of J presented in [2] by using averaging approximations, eight basis functions of linear Legendre multi-wavelets in [14] and hybrid function approximation method in [11]. The numerical results for the optimal state trajectories and optimal control are displayed in Figures 1-3.

Table 1: Simulation results of Example 1

N	J_N	$\left\|\frac{J_N - J_{N-1}}{J_N}\right\|$
5	3.3971	-
6	3.3990	6.472×10^{-4}
7	3.3991	2.941×10^{-5}

Example 2. Consider the optimal control problem of a harmonic oscillator with retarded damping as in [2]:

Table 2: The cost functional values for Example 1

Method	Cost functional values
Banks et al. [2]	3.2587
Kellat [14]	3.43254
Haddadi et al. [11]	3.21663
Present method	3.3991

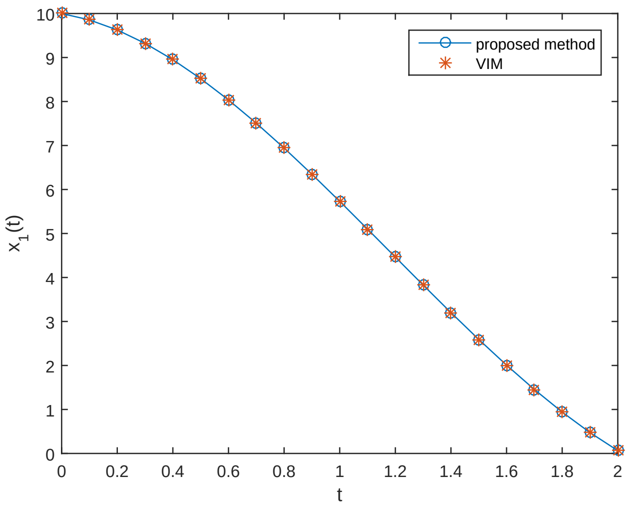

Figure 1: The suboptimal state, when $N = 7$ for Example 1

$$J = \frac{1}{2}(x_1^2(t) + x_2^2(t)) + \frac{1}{2}\int_0^2 u^2 dt, \tag{27}$$

subject to:

$$\begin{cases} \ddot{x}(t) + \dot{x}(t) + x(t-1) = u(t), \\ x(0) = 1, \quad \dot{x}(0) = 0, \end{cases} \tag{28}$$

where $0 \leq t \leq 2$. equation (28) equivalent to

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -x_1(t-1) - x_2(t) + u(t), \\ x_1(0) = 1, \quad x_2(0) = 0. \end{cases} \tag{29}$$

The problem is to find the control vector $u(t)$ which minimizes equation (27) subject to equation(29). The exact solution for this problem is given by [2]:

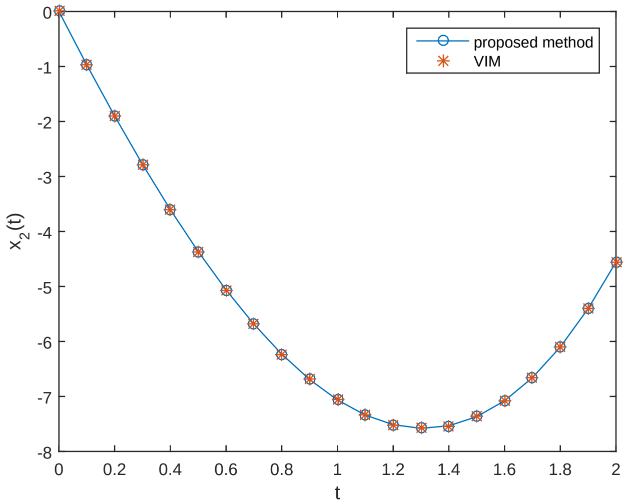

Figure 2: The suboptimal state, when $N = 7$ for Example 1

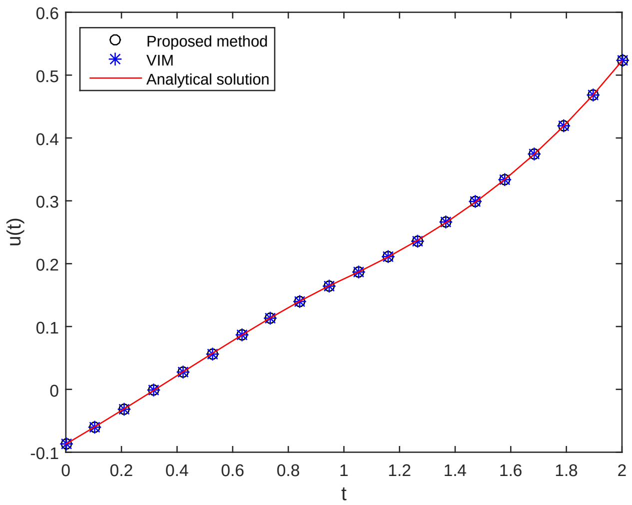

Figure 3: The suboptimal control, when $N = 7$ for Example 1

$$u^*(t) = \begin{cases} (\mu - \delta)e^{t-2} + [2\mu - 3\delta - (\mu - \delta)t]e^{t-1} + \delta(t + 2) - \mu & 0 \leqslant t < 1, \\ (\mu - \delta)e^{t-2} + \delta \quad 1 \leqslant t \leqslant 2, \end{cases}$$

where $\mu = 0.5226194$, $\delta = -0.0259256$, and the optimal cost functional is $J^* = 0.197478$. Implementing the algorithm described in section 4 with the tolerance error bounds $\epsilon = 2 \times 10^{-5}$, the desired suboptimal control is obtained for $N = 8$ iterations. From Table 3, it is observed

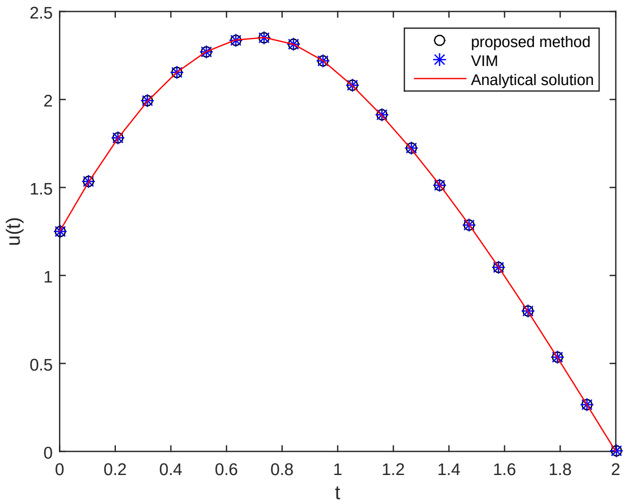

Figure 4: The suboptimal control, when $N = 8$ for Example 2

that $\left|\dfrac{J_8 - J_7}{J_8}\right| = 1.012 \times 10^{-5} < 2 \times 10^{-5}$, and a minimum value $J_8 = 0.197473$ is obtained. A comparison is made between the value of J obtained by the present method and $N = 8$, with the value $J = 0.1934$ presented in [2] by using averaging approximations. The numerical results for the approximate solution of $u(t)$ by using the HPM and VIM and the exact solution are graphically the same as shown Figure 4.

Table 3: Simulation results of Example 2

N	J_N	$\left\|\dfrac{J_N - J_{N-1}}{J_N}\right\|$
6	0.197452	-
7	0.197471	9.621×10^{-5}
8	0.197473	1.012×10^{-5}

6 Conclusions

In the present work, a technique was developed for obtain the optimal solution of the controlled harmonic oscillator. We described the method and used it in some test examples in order to show its applicability and validity in comparison with other methods and exact solutions. We achieved satisfactory approximations with a few number of iterations, revealing the efficiency of the method. Moreover, since this method does not need the discretization of the variables, there is no computation round off errors and one is not faced with the necessity of large computer memory and time.

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حل عددی نوسانگر هارمونیک کنترل شده با استفاده از روش اختلال هموتوپی

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چکیده

مسأله نوسانگر هارمونیک کنترل شده با دمپینگ، دسته‌ی مهمی از مسائل کنترل بهینه است، که نقش مهمی در زمینه پدیده نوسانی در سیستم‌های مهندسی غیرخطی ایفا می‌کند. در این مقاله، برای حل این مسأله، روش تحلیلی ارائه می‌شود. این رویکرد بر اساس روش اختلال هموتوپی پایه ریزی شده است. فرایند حل آسان و ساده است. برای این منظور، یک الگوریتم طراحی کنترل با پیچیدگی محاسبات کم، پیشنهاد می‌شود. هم چنین، به کمک تکرارهای متناهی از الگوریتم پیشنهادی، یک قانون کنترل زیر بهینه برای مسأله به دست می‌آید. در نهایت، نتایج به دست آمده با جواب دقیق مسأله نوسانگر هارمونیک و سایر نتایج حاصله از آثار قبلی مقایسه شده، که به وضوح، دقت بالای نتایج آشکار است.

کلمات کلیدی

کنترل زیر بهینه، نوسانگر هارمونیک، دمپینگ، روش اختلال هموتوپی .