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# Optimal Adaptive Sliding Mode Control for a Class of Nonlinear Affine Systems

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**Abstract.** This paper presents an optimal robust adaptive technique for controlling a certain class of uncertain nonlinear affine systems. The proposed approach combines sliding mode control, a linear quadratic regulator for optimality, and gradient descent as an adaptive controller. The convergence of the sliding mode control process is proven using two theorems based on the Lyapunov function. Simulation results for pendulum and inverted pendulum systems demonstrate that the proposed method outperforms both the linear quadratic regulator technique and the sliding mode control regarding reduced chattering and improved reaching time.

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## 1 Introduction

In the literature, general nonlinear dynamic systems are studied in two separate fields: qualitative analysis and controller design. The latter involves achieving synchronizing, stabilizing, and tracking the desired signals. However, due to the presence of uncertainty and external disturbances, practical problems often require complex control design. To address this, various control methods have been proposed nonlinear systems, including adaptive control [35], fuzzy control [19], sliding mode control [12], and feedback linearization [19]. In this paper, we focus on the controlling and stabilizing a class of uncertain nonlinear affine systems described by the following dynamic system [15]:

$$
\begin{cases}
\dot{x}_1 = x_2, \\
\dot{x}_2 = x_3, \\
\vdots \\
\dot{x}_n = f(x) + g(x)u + d(t),
\end{cases}
\tag{1}
$$

where $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the input value, $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are smooth functions, and $d(t) \in \mathbb{R}$ represents external disturbances. Furthermore, we assume that, $d(t)$ is bounded, $|d(t)| \leq \gamma$ , where $\gamma$ is a positive real number. System (1) has many important applications in engineering and diseases [11, 39].

Real-world processes often involve uncertainty and external disturbances, which require controllers that can eliminate disturbances and be robust to uncertainty. One such method is sliding mode control (SMC), which has gained popularity due to its robustness to disturbances, parameter changes, and noise. SMC has been successfully applied in various nonlinear systems, including spacecraft [1], power converters [4], robotics [37], and diseases [18]. Previous works have proposed different types of SMC techniques, such as SMC with an integral sliding surface was proposed for classes of uncertain nonlinear systems by Mirhosseini-Alizamini et al [7, 8, 9, 13]. Mahmoodabadi and Soleymani [19] divided the fourth-order nonlinear system into two separate subsystems with one input and applied the decoupled sliding surface technique. Liu and Wang [17] used linear matrix inequalities to define the sliding surface for the inverted pendulum system.

In spite of robust control methods, optimal control methods are often preferred for designing controllers with minimum cost. Optimal control techniques have been applied in diverse fields, including medicine [31] and engineering [24]. For linear systems, the linear quadratic regulator (LQR) technique is widely used due to its simplicity and systematic design structure [16]. However, designing an optimal controller for a nonlinear system is more complex than for linear system. Various methods have been proposed in the literature. for controlling nonlinear systems optimally, Such as Pontryagin minimum principle [22, 23], Hamilton–Jacobi–Belman [10] and the optimal control based on state-dependent Riccati equation [27].

The integration of the optimal control and SMC techniques can yield an optimal version of SMC, referred to as OSMC, for uncertain systems. This combination has been explored in various studies. For instance, in [21], the combination of LQR and integral SMC proved to be more efficient effective than conventional methods. A hybrid controller was developed for the rotary inverse pendulum, leveraging the optimality of the LQR controller and robustness of SMC while eliminating the no robust arrival phase in SMC [3]. Soon et al. integrated and optimized a sliding controller with a proportional-integral-derivative (PID) controlle [34]. Kumar and Mija [14] proposed LQR and SMC methods for a class of nonlinear systems represented in a cascade form. Sanjeewa and Parnichkun [32] attempted to design LQR based on SMC for nonlinear systems with time-varying delays and uncertainties.

Although OSMC has advantages, it is susceptible to chattering, which can result in thermal losses, mechanical depreciation, and the excitation of high-frequency dynamics that are not accounted for in the model. Numerous methods have been proposed to address this issue, including high-order SMC [25, 29] and the use of continuous approximations such as the saturation function in the control design.

Adaptive controllers are also utilized when system parameters are unknown or change over time. Plestan and Shtessel [28] developed two adaptive SMC (ASMC) methods for controlling nonlinear systems with bounded uncertainties of unknown boundaries. An adaptive OSMC (OASMC) based on a disturbance observer (DOB) has been the employed to optimize and stabilize a time-varying manipulator robot system with uncertainty [5],where an online DOB estimates unknown disturbances to adjust the control gains. Similar approaches have been used to control chaotic oscillations in a seven-dimensional power system [36]. Mahmoodabadi et al. presented an adaptive robust PID SMC for a liquid-level system [20]. Bkekri et al. [2] proposed ASMC for the knee joint orthopedic protection system. Ranjbar et al. [30] have suggested a robust ASMC for a micro electromechanical system capacitor that the regulated output is accurate and stable in the face of disturbance and uncertainty. An optimal fuzzy combination of discrete SMC (DSMC) and adaptive feedback linearization has been proposed and compared for nonlinear systems with uncertainty [19]. Additionally, Vaseei and Zarrabi [35] proposed a new Lyapunov-based adaptive controller for the AIDS virus.

Based on above discussion, the advantages of OSMC and ASMC have been widely recognized, but the combination of SMC, adaptive control and optimal control has not been extensively explored. In previous studies, adaptive techniques were primarily utilized to estimate the unknown upper bound of uncertainties. In this study, we propose an OASMC technique for a class of uncertain nonlinear systems. We leverage the optimality in the LQR controller and robustness of the sliding mode controller. To reduce chattering, we apply adaptive control to estimate the sliding gain and the thickness of the boundary layer. Specifically, we design an LQR based adaptive sliding mode controller for pendulum and inverted pendulum systems with uncertainty. Simulation results demonstrate the stability of this method and its ability to reduce chattering compared to SMC.

This paper is structured into four sections. Section 2 discusses outlines the OASMC approach. Section 3 presents simulation results for pendulum and inverted pendulum systems with uncertainty. Finally, Section 4 provides conclusions based on our findings.

## 2 The OASMC Approach

In this section, we employ an OASMC approach to ensure the stability of the nonlinear system (1). First step involves designing a standard SMC to eliminate external disturbance. Subsequently, an LQR controller is incorporated to minimize the cost function. Finally, an adaptive strategy is utilized to determine the sliding gain and the thickness of the boundary layer.

### 2.1 The SMC method

The SMC is a well-known controller that exhibits high sensibility to reduce errors and produces fast control responses. This control strategy has been widely used in various industrial, medical, and economic systems. A range of sliding surfaces is employed to eliminate disturbance and uncertainty. Here, we utilize the standard sliding surface.

Consider the uncertain nonlinear system (1) with the relative degree of $n$. The standard sliding surface $s(t)$ is defined as [18]

$$s(t) = (\frac{d}{dt} + \lambda)^{n-1} e_1(t), \tag{2}$$

where $\lambda$ is the weighting coefficient of the state variable errors and $e$ represents the state variable error of $x$ as follows:

$$e(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_n(t) \end{bmatrix} = \begin{bmatrix} x_1 - x_{1d} \\ x_2 - x_{2d} \\ \vdots \\ x_n - x_{nd} \end{bmatrix}, \tag{3}$$

in which $x_d$ is the desired state signal. The SMC law has two parts:

$$u_{SMC} = u_{eq} + u_{sw}, \tag{4}$$

where $u_{eq}$ is a continuous control that results in the system reaching the sliding surface and it is obtained by solving $\dot{s} = 0$. Additionally, $u_{sw}$ is an almost continuous control that is added to overcome the uncertainty terms and maintain motion on the sliding surface. In special case, we set $u_{sw} = k_1 sgn(s)$, where $k_1$ is a positive constant.

**Theorem 1.** Consider the system (1) without uncertainty, where the function $g(x)$ is invertible. Let the sliding surface (2) be defined, and let the control law is described as follows:

$$\begin{gathered} u_{SMC} = g^{-1}(x)\{\dot{x}_{nd} - f(x) - \sum_{i=0}^{n-1} c_i e_1^{(i)}(t)\} - k_1 \text{sgn}(s), \\ k_1 > 0, \quad c_i = \binom{n-1}{i} \lambda^{n-1-i}. \end{gathered} \tag{5}$$

Then, the tracking error trajectory will converge to the sliding surface (2) within a finite time.

*Proof.* From (2) and the Newton's binomial expansion, we can obtain:

$$s(t) = \sum_{i=0}^{n-1} c_i e_1^{(i)}(t), \quad c_i = \binom{n-1}{i} \lambda^{n-1-i}. \tag{6}$$

where $e_1^{(i)}$ is the $i$ th derivative of $e_1$. Choosing the Lyapunov function as:

$$V = \frac{1}{2} s^2. \tag{7}$$

By differentiating Equation (7), we obtain:

$$\dot{V} = s\dot{s} = s(\sum_{i=0}^{n-1} c_i e_1^{(i)}(t) + f(x) + g(x) u_{SMC} - \dot{x}_{nd}). \tag{8}$$

Substituting (5) into (8), we obtain:

$$\dot{V} = (-k_1 \text{sgn}(s))s = -k_1|s| \leq 0. \tag{9}$$

Therefore, based on the Lyapunov stability theorem, the state variables will converge to the equilibrium point. From equations (7) and (9), we can derive

$$\dot{V} = \dot{s} = \frac{ds}{dt} = -k_1 \text{sgn}(s), \tag{10}$$

which leads to

$$dt = -\frac{d|s|}{k_1}. \tag{11}$$

Now, it can be proven that $s(t)$ reaches to zero in a finite time such as $T^*$. In this case $s(T^*) = 0$. FTo do that, assuming $s(0) \neq 0$, we integrate both sides of (11) from 0 to $T^*$

$$\int_0^{T^*} dt = -\frac{1}{k_1} \int_{|s(0)|}^{|s(T^*)|} dt, \tag{12}$$

which is equivalent to (13):

$$T^* = \frac{|s(0)|}{k_1}. \tag{13}$$

Therefore, the tracking error trajectory of system (1) will converge to the sliding surface $s$ within the finite time $T^*$, and the proof is complete. □

**Theorem 2.** Consider system (1) with uncertainty. The control law (5) with the conditions presented in Theorem 1, and furthermore, $k_1 \geq \gamma$, guarantees convergence of the tracking error trajectory to the sliding surface (2) in a finite time.

*Proof.* By differentiating Equation (6), we obtain:

$$\dot{s} = \sum_{i=1}^{n-1} c_i e_1^{(i)}(t) + f(x) + g(x)u + d(t) - \dot{x}_{nd}. \tag{14}$$

By substituting (5) into (14), we have:

$$\dot{s} = d(t) - k_1 \text{sgn}(s). \tag{15}$$

Now, let the Lyapanov function as follows:

$$V = \frac{1}{2}s^2. \tag{16}$$

Using (15) and $|d(t)| \leq \gamma$, we have:

$$\dot{V} = s\dot{s} \leq s(k_1(1 - \text{sgn}(s))). \tag{17}$$

Here, $k_1 \geq \gamma$ is a constant parameter. According to the sign of $s$ we have $\dot{V} \leq 0$. The rest of the proof is similar to that of Theorem 1. □

**Remark 1.** The Lyapunov function $V$, used in Theorem 2, guarantees that system (1) is asymptotically stable, ensuring that all trajectories starting from outside the sliding surface reach the sliding surface $s = 0$ in a finite time $T^*$. Once on the sliding surface, $s(t)$ remains zero for all $t > T^*$.

An optimal control scheme can be used to reduce the cost of control design. To do that, firstly, the system (1) is linearized, and then a nominal LQR controller is designed.

**Remark 2.** The LQR technique is an optimal control method for linear systems with a quadratic objective function. Assuming a state space representation of the nominal linearized system (1) without disturbance, is given as follows:

$$\dot{x} = Ax(t) + Bu(t), \tag{18}$$

where $x(\cdot) \in \mathbb{R}^n$ is the state vector, $u(\cdot) \in \mathbb{R}$ is the control value, $A \in \mathbb{R}^{n\times n}$ is the state matrix, and $B \in \mathbb{R}^{n\times 1}$ is the control matrix. The objective is to find the optimal control such that the square cost function [6],

$$J = \int_{t_0}^{t_1} (x^T(t)Qx(t) + u^T(t)Ru(t))dt, \tag{19}$$

is minimized, wherein $Q$ is an $n \times n$ semi-positive definite matrix and $R$ is an $m \times m$ positive definite matrix. The optimal control $u_{LQR}$ is then calculated as follows:

$$u_{LQR} = -R^{-1}B^T Px(t) = -kx(t), \tag{20}$$

where $k = R^{-1}B^T P$ and $P$ is a symmetric and positive definite matrix that is the solution of the following Riccati algebraic equation:

$$PA + A^T P - PBR^{-1}B^T P + Q = 0. \tag{21}$$

To deal with uncertainties and minimize control input, the sliding mode controller (4) is integrated with LQR controller.

$$u(t) = u_{LQR}(t) + u_{SMC}(t), \tag{22}$$

where $u_{LQR}(t)$ is the optimal control applied to the nominal system and $u_{SMC}(t)$ is the input control of the SMC method to deal with uncertainties.

### 2.2 Gradient descent method

The primary concept behind adaptive control is to adjust control parameters using a suitable mechanism. When the parameters of a system are either unknown or vary with time, adaptive controllers are employed. The gradient descent method [33], as is a suitable adaptation law used to update the parameters of the OSMC. The key idea is to update the parameter in the opposite direction of the cost function. Mathematically speaking, let $J : \mathbb{R}^n \to \mathbb{R}$ represent the desired cost function and $\theta^*$ denote its minimum vector. The gradient descent method involves iteratively updating the initial approximation $\theta_0$ for the minimum $\theta^*$ to the subsequent points $\theta_1, \theta_2, \ldots$ in $\mathbb{R}^n$ in the iterative method, $\theta_{k+1} = \theta_k - h\nabla J(\theta_k)$, until the stopping condition is met. Where $h$ is represents the step length. Assuming an infinitesimal step length, the previous equation can be expressed as $\dot{\theta} = -h\nabla J(\theta)$.

As a result, the control gains are adjusted using the sliding surface to mitigate chattering. Hence, to reduce chattering, the boundary layer method and the saturation function $(\text{sat}(\cdot))$,are employed, with the sign function in the control design. Consequently, Equation (5) is modified as shown in [19],

$$u_{SMC} = u_{eq} - k_1 \text{sat}(\frac{s}{\phi}), \tag{23}$$

where $s$ is the sliding surface, $k_1$ is the sliding gain, and $\phi$ is the thickness of the boundary layer. Furthermore, the saturation function is defined as

$$\text{sat}(\frac{s}{\phi}) = \begin{cases} -1, & \frac{s}{\phi} \leq -1, \\ \frac{s}{\phi}, & -1 < \frac{s}{\phi} < 1, \\ 1, & \frac{s}{\phi} \geq 1. \end{cases} \tag{24}$$

Considering Equation (23), the chain derivative rule is applied to update $k_1$ and $\phi$ as follows [18]:

$$\dot{\phi} = -\psi_1 \frac{\partial s\dot{s}}{\partial \phi}, \tag{25}$$

and

$$\dot{k}_1 = -\psi_2 \frac{\partial s\dot{s}}{\partial k_1}, \tag{26}$$

where $\psi_1$ and $\psi_2$ are the learning rates, and they are positive constant parameters.

## 3 Simulation Results

This section presents the simulation results obtained by employing the OASMC technique to simulate the pendulum and inverted pendulum systems. Furthermore, to demonstrate the effectiveness of this approach, a comparison is made with the SMC and the LQR.

### 3.1 The OASMC of pendulum

The uncertain nonlinear system of the pendulum can be described as [15]:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{b}{lm^2}x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d(t), \tag{27}$$

where $m$ is the mass of the pendulum, $l$ is the length of the pendulum, $b$ is the coefficient of friction, $g$ is the acceleration due to gravity, and $d(t)$ represents disturbances.

By employing the Jacobian linearization method around the equilibrium point $[0, 0]^T$, the state equations of the linearized system with disturbances are obtained as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{b}{lm^2} \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d(t). \tag{28}$$

The tracking error is

$$e = x - x_d,$$

or

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} x_{1d} \\ x_{2d} \end{bmatrix}, \tag{29}$$

where $x_d$ represents the desired state to be tracked. The sliding surface is defined as follows:

$$s = e_2 + \lambda e_1. \tag{30}$$

Taking the derivative of Equation (30), we obtain

$$\dot{s} = \dot{e}_2 + \lambda \dot{e}_1 = 0. \tag{31}$$

Substituting (28) and (29) into (31), we get

$$\dot{s} = -\frac{g}{l}\sin x_1 - \frac{b}{ml^2}x_2 + d(t) + u_{eq} - \dot{x}_{2d} + \lambda(x_2 - x_{2d}) = 0. \tag{32}$$

Therefore, the equivalent control input $u_{eq}$ is given by

$$u_{eq} = \frac{g}{l}\sin x_1 + \frac{b}{ml^2}x_2 + \dot{x}_{2d} - \lambda(x_2 - x_{2d}). \tag{33}$$

Based on Equation (23), the sliding mode control input $u_{SMC}$ is expressed as

$$u_{SMC} = \frac{g}{l}\sin x_1 + \frac{b}{ml^2}x_2 + \dot{x}_{2d} - \lambda(x_2 - x_{2d}) - k_1 \mathrm{sat}(\frac{s}{\phi}). \tag{34}$$

To estimate the controller gain in (34), $k_1$ and $\phi$, using gradient descent method, we have

$$\dot{\phi} = -\psi_1 \frac{\partial s\dot{s}}{\partial \phi}. \tag{35}$$

Using the chain derivative rule, (35) is reformulated as

$$\dot{\phi} = -\psi_1 (s\frac{\partial \dot{s}}{\partial u_{SMC}} + \dot{s}\frac{\partial s}{\partial u_{SMC}})\frac{\partial u_{SMC}}{\partial \phi}. \tag{36}$$

Accordingly, from the above equations, we have

$$\frac{\partial s}{\partial u_{SMC}} = 0, \quad \frac{\partial \dot{s}}{\partial u_{SMC}} = 1, \quad \frac{\partial u_{SMC}}{\partial \phi} = \begin{cases} k_1 \frac{s}{\phi^2}, & |\frac{s}{\phi}| < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{37}$$

Therefore, from (36) and (37), $\dot{\phi}$ is obtained as

$$\dot{\phi} = \begin{cases} -\psi_1 k_1 \frac{s^2}{\phi^2}, & |\frac{s}{\phi}| < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{38}$$

Additionally, we have

$$\dot{k}_1 = -\psi_2 \frac{\partial s\dot{s}}{\partial k_1}. \tag{39}$$

According to the aforementioned process, we can calculate $\dot{k}_1$ as:

$$\dot{k}_1 = -\psi_2(s\frac{\partial \dot{s}}{\partial u_{SMC}} + \dot{s}\frac{\partial s}{\partial u_{SMC}})\frac{\partial u_{SMC}}{\partial k_1}. \tag{40}$$

Hence,

$$\dot{k}_1 = \psi_2 \text{sat}(\frac{s}{\phi}). \tag{41}$$

The system parameters are assigned the following values: $g = 10$, $l = 1$, $m = 1$, $b = 2$ [15]. To perform numerical simulation, we consider the disturbance $d(t) = 2\sin t$. Additionally, the initial conditions of the system and the desired state for tracking the system are set as $x(0) = [-1, 1]^T$, $x_d = [0, 0]^T$, respectively. We assume $k_1 = 2$. Furthermore, we set $Q = 3I_2$ and $R = [1]$. By utilizing Equation (20), we obtain $k = [0.1489 \quad 0.7014]$. Figure 1 illustrates the sliding surface of pendulum, both in the absence and presence of the disturbance. The finite-time reaching law holds for the specified sliding surface. Figures 2, 3, and 4 depict the simulation results of control input and position tracking of LQR, SMC, and OASMC, respectively. As observed, LQR is unstable in the presence of the sinusoidal signal, whereas SMC and OASMC remain stable. Moreover, OASMC achieves a faster reaching time than SMC. In Figure 3(a), it can be seen that SMC exhibits significant control chattering [33]. Consequently, the OASMC approach outperforms LQR and SMC in terms of performance.

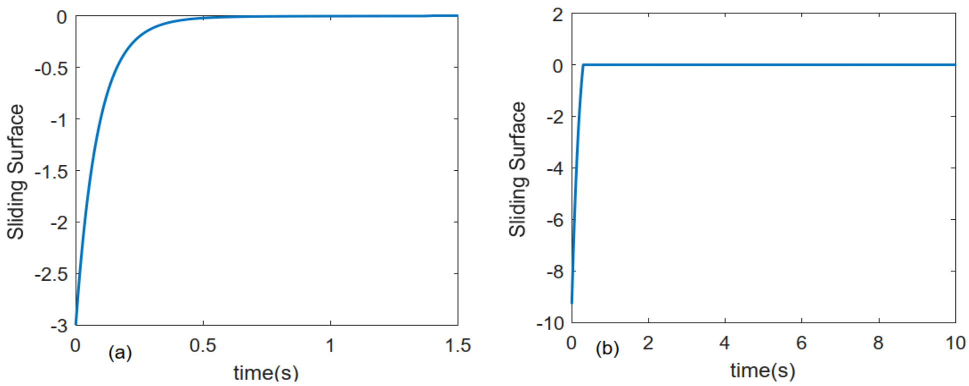


**Figure 1:** Sliding surface of the pendulum system (a): without a disturbance, (b) with a disturbance.

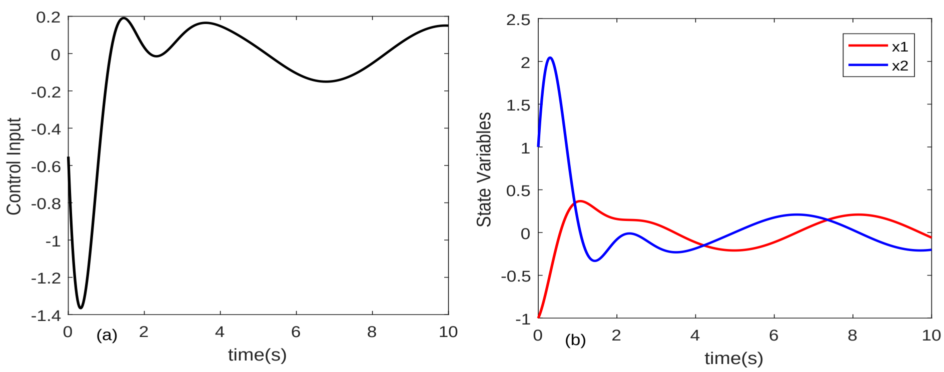


**Figure 2:** Simulation results for control input and state variables of LQR in the pendulum system.

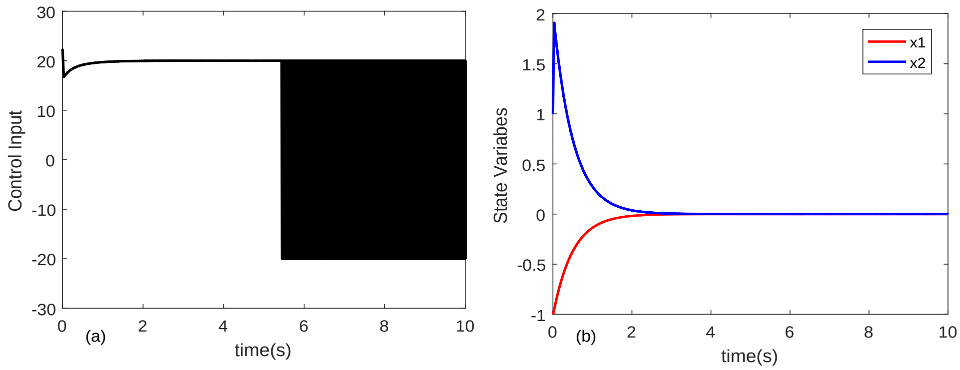


**Figure 3:** Simulation results for the control input and state variables of SMC in the pendulum system.

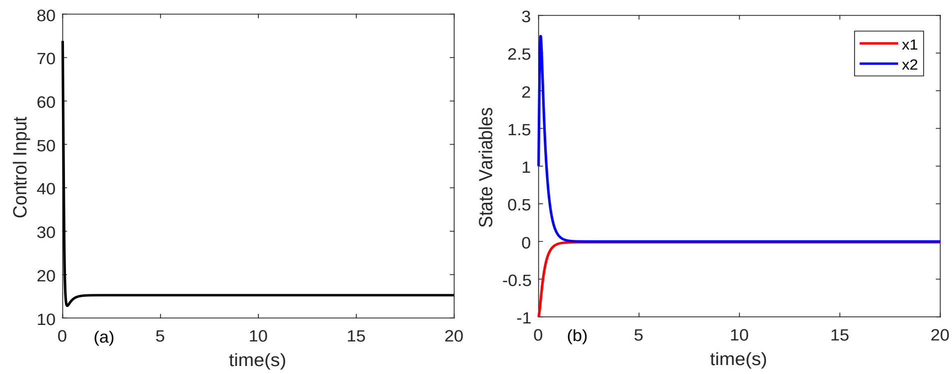


**Figure 4:** Simulation results for the control input and state variables of OASMC in the pendulum system.

### 3.2 The OASMC of inverted pendulum

The inverted pendulum system can be described by the following state space equations [38]:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{-mg \cos x_3 \sin x_3 + ml \sin x_3 x_4^2}{M + m \sin^2 x_3} \\ x_4 \\ \frac{-ml \cos x_3 \sin x_3 x_4^2 + (M+m) g \sin x_3}{Ml + ml \sin^2 x_3} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M + m \sin^2 x_3} \\ 0 \\ \frac{1}{Ml + ml \sin^2 x_3} \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} d(t), \tag{42}$$

where $g = 9.8$, $M$ is the mass of the vehicle, $m$ is the mass of the pendulum, $L$ is the length of the pendulum, $l = \frac{1}{2}L$, and $u$ is the control input. By applying the Jacobian linearization method around the equilibrium point $[0, 0, 0, 0]^T$, the state equations of the linearized system with disturbance can be expressed as:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a_1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ a_4 \\ 0 \\ a_3 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} d(t), \tag{43}$$

where,

$$a_1 = \frac{m(m+M)gl}{(m+M)I + mMl^2}, \qquad a_2 = -\frac{m^2gl^2}{(m+M)I + mMl^2},$$

$$a_3 = -\frac{ml}{(m+M)I + mMl^2}, \qquad a_4 = \frac{I + ml^2}{(m+M)I + mMl^2},$$

$$I = \frac{1}{12}mL^2 \quad \text{and} \quad |d(t)| \leq \gamma.$$

According to [26], the system represented in Equation (43) can be transformed into the following cascaded form using a coordinate transformation.

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = f_1(x_1, x_2, x_3, x_4), \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = f_2(x_1, x_2, x_3, x_4) + g(x_1, x_2, x_3, x_4)u + d(t). \end{cases} \tag{44}$$

Assuming that $f_1(x_1, x_2, x_3, x_4) = 0$ at the equilibrium point and $\frac{\partial f_1}{\partial x_3}$ is invertible, the error dynamics of the system are defined as

$$\begin{cases} e_1 = x_1, \\ e_2 = x_2, \\ e_3 = (a_2 - \frac{a_1 a_4}{a_3})x_3, \\ e_4 = (a_2 - \frac{a_1 a_4}{a_3})x_4. \end{cases} \tag{45}$$

The sliding surface is defined as:

$$s = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + e_4, \tag{46}$$

where $\lambda_i$ $(i = 1, 2, 3)$ are positive constants. The SMC law is obtained as:

$$u_{SMC} = -\Big[\frac{1}{a_2 - \frac{a_1 a_4}{a_3}}\Big]^{-1} \{\lambda_1 x_2 + \lambda_2 (a_2 - \frac{a_1 a_4}{a_3})x_3 + \lambda_3 \big(a_2 - \frac{a_1 a_4}{a_3} x_4\big) \tag{47}$$

$$+ (a_2 - \frac{a_1 a_4}{a_3})d(t) + k_1 \text{sat}(\frac{s}{\phi})\}. \tag{48}$$

Additionally, $\phi$ and $k$ are updated according to the following formulas:

$$\dot{\phi} = \begin{cases} -\psi_1 k_1 (a_2 - \frac{a_1 a_4}{a_3}) \frac{s^2}{\phi^2}, & |\frac{s}{\phi}| < 1, \\ 0, & \text{otherwise}, \end{cases} \tag{49}$$

and

$$\dot{k} = \psi_2 s (a_2 - \frac{a_1 a_4}{a_3}) \text{sat}(\frac{s}{\phi}). \tag{50}$$

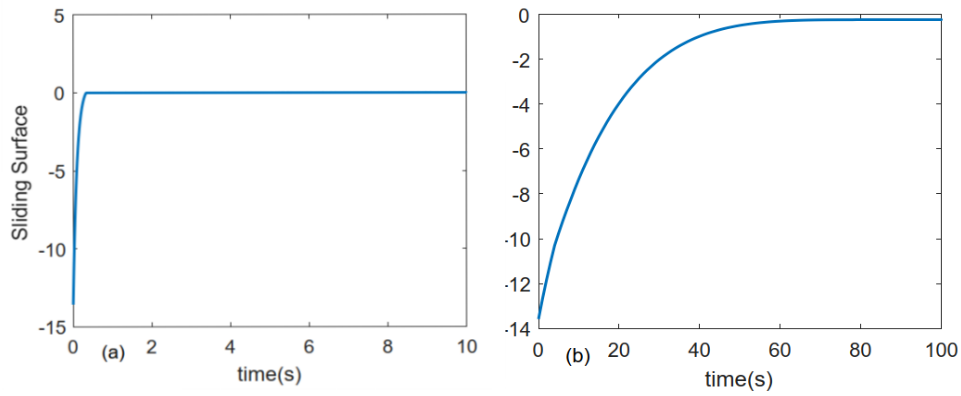


**Figure 5:** Sliding surface of the inverted pendulum (a): without disturbance, (b): with disturbance.

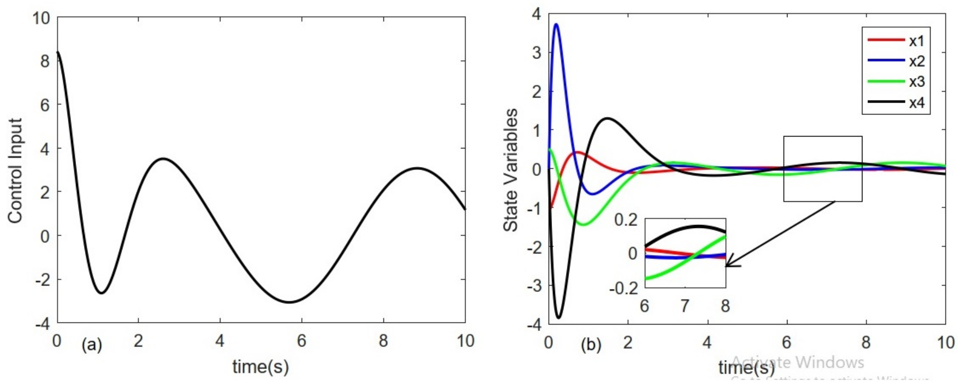


**Figure 6:** Simulation of control input and state variables of LQR for the inverted pendulum system.

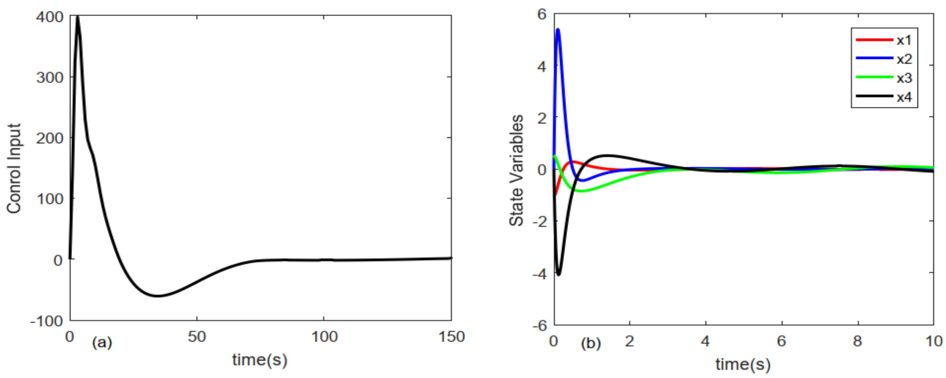


**Figure 7:** Simulation of control input and state variables of SMC for the inverted pendulum system.

To simulate the system, we consider the following parameter values: $g = 9.8$, $M = 1$, $m = 0.1$, $L = 0.5$, $x(0) = \left[-\frac{\pi}{3}, 0, 5, 0\right]^T$ [14], $x_d = [0, 0, 0, 0]^T$, $d(t) = 0.2 \sin t$, $k_1 = 0.2$, $Q = I_2$, $R = [1]$. From Equation (20), we obtain $k = \begin{bmatrix} -28.6767 & -5.2444 & -1.0000 & -2.1569 \end{bmatrix}$. Figure 5 shows the sliding surface of the inverted pendulum, both without disturbance and with disturbance. The simulation

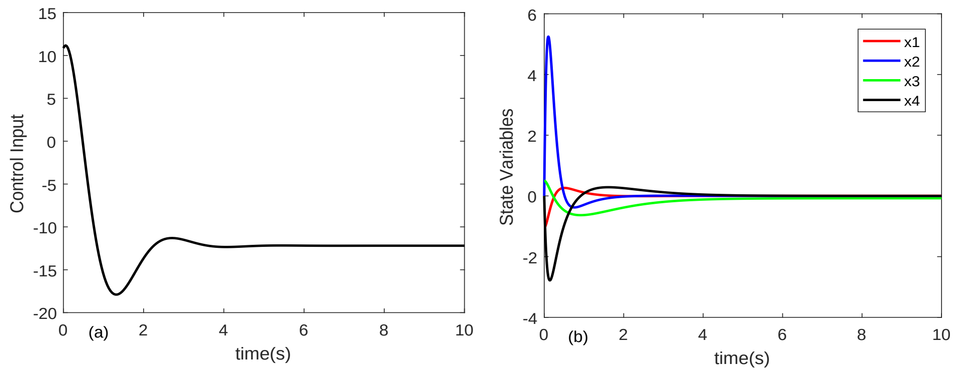


**Figure 8:** Simulation of control input and state variables of OASMC for the inverted pendulum system.

results of the control input and position tracking of the inverted pendulum are shown in Figures 6–8. As shown in Figure 6, the LQR is unstable. On the other hand, as seen in Figures 7 and 8, the SMC and the OASMC are stable, and the convergence of the state variables is desirable. Furthermore, OASMC outperforms SMC in terms of performance.

## 4 Conclusion

This study introduced an approach that combines the Linear Quadratic Regulator (LQR) with a sliding mode controller and an adaptive controller for a class of uncertain nonlinear systems. The nonlinear system was first linearized, followed by the integration of the LQR technique with the SMC method. Additionally, the gradient descent method was utilized to mitigate chattering. Simulation results demonstrate that the proposed method exhibits superior robust performance compared to the LQR and the SMC methods. Furthermore, the control input of this method effectively addresses the chattering issue compared to the SMC. By considering the system without linearization and incorporating optimal nonlinear control, further advancements can be made based on the results.

## Declarations

### Availability of Supporting Data

All data generated or analyzed during this study are included in this published paper.

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### Competing Interests

The authors declare that they have no competing interests relevant to the content of this paper.

**Authors' Contributions**

The main text of manuscript is collectively written by the authors.

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