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Model Predictive Control for a 3D Pendulum on $SO(3)$ Manifold Using Convex Optimization

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Abstract. Conventional model predictive control (MPC) methods are usually implemented to systems with discrete-time dynamics laying on smooth vector space $\mathbf{R}^n$. In contrast, the configuration space of the majority of mechanical systems is not expressed as Euclidean space. Therefore, the MPC method in this paper has developed on a smooth manifold as the configuration space of the attitude control of a 3D pendulum. The Lie Group Variational Integrator (LGVI) equations of motion of the 3D pendulum have been considered as the discrete-time update equations since the LGVI equations preserve the group structure and conserve quantities of motion. The MPC algorithm is applied to the linearized dynamics of the 3D pendulum according to its LGVI equations around the equilibrium using diffeomorphism. Also, as in standard MPC algorithms, convex optimization is solved at each iteration to compute the control law. In this paper, the linear matrix inequality (LMI) is used to solve the convex optimization problem under constraints. A numerical example illustrates the design procedure.

Keywords. Model predictive control, Convex Optimization, Linear matrix inequality, Lie group variational integrator.

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1 Introduction

Model Predictive Control (MPC) is based on predicting the behavior of a system using its dynamical model and optimizing the prediction to have the best decision. Due to this reason, the dynamic models of the system play an essential role in solving MPC problems. A control system is considered as a family of vector fields, and the dynamical system is the flow generated by this vector field [1, 2]. Differential geometry is a new language of Lagrangian and Hamiltonian Mechanics. In differential geometry, the state space of the system is modeled as manifolds, which are locally diffeomorphic with Euclidean space. As a result, this method is a coordinate-free method that applies to infinite-dimensional systems. Since mechanical systems are symmetric, the states of the system do not change under a certain transformation; this criterion is expressed by Lie group actions [3]. A Lie group that is a smooth manifold with a group structure, is a mathematical concept appropriate for describing continuously varying groups of transformation [3]. In [4], geometric mechanics of rigid bodies on a Lie group is expressed based on the Euler/Lagrange equation of mechanical systems that are developed according to Hamilton's principle. A so-called Lie Group Variational Integrator (LGVI) method has been produced for systems with a Lie group configuration space. The main target of LGVI is implementing an exponential map representing the variation of a curve on a Lie group in terms of Lie algebra element. This method is developed to acquire the discrete-time dynamic equations of the system that preserves the Lie group structure. The main result of this method is that the achieved update discrete-time equations are coordinate-free, namely, there is no need to choose a specific local coordinate. This totally avoids ambiguity and singularity associated with local coordinates [4, 5]. Considering the LGVI method to model the discrete-time update equations of motions, preserves the conserved quantities of motion and therefore provides a more realistic prediction model. As other standard integrating methods such as Runge-Kutta do not use the group structure so that they are deprived of this property. The LGVI method updates the rotational matrix by multiplying two matrices in $SO(3)$, which guarantees the rotational matrix still remains on $SO(3)$ and preserves the conservative motion. [7, 6] provide other types of variational integrator methods. The conventional MPC methods are usually applied to systems with discrete dynamics on $\mathbf{R}^n$ vector space. However, the configuration space of the majority of systems is smooth manifolds which are not diffeomorphic to $\mathbf{R}^n$. For designing the predictive dynamics of such systems, the manifolds with limited dimensions are embedded in $\mathbf{R}^n$, then standard integrating methods are implemented until the discrete updating equations are achieved. Different methods of integrating system dynamics on manifolds have been developed, see [7, 8, 9] for example. Development of the model predictive control design for dynamics evolving on smooth manifolds is considered in [10, 11]. The method of linearizing and embedding the system in $\mathbf{R}^n$ has been used in these papers. Implementing the MPC-based LMI approach is a technique for controlling plants with uncertainties. Since the optimization-based LMI method can be solved in polynomial time, it is applicable to implement it in on-line optimization problems [12]. Solving an optimization problem at each sampling time within a receding horizon is the main contribution of MPC algorithms so that the development of optimization methods improves the ability to solve MPC problems. In such issues, an optimization problem that is efficiently solvable via linear matrix inequality (LMI) can be extended to MPC. In this case, a min-max optimization is solved, which computes the control law by minimizing a quadratic cost function subject to constraints in worst-case at each time step. The issue of solving a min-max optimization can be considered as a convex optimization with linear matrix inequality. Besides, using the LMI optimization scheme with MPC at each time instant can incorporate uncertainties as input and output constraints, and guarantees the robustness properties of the system at the same time. Since Lie group variational integrator is one of the rigid body computational methods that maintain Lagrangian/Hamilton structures as well as the structure of rigid body configuration group, in this paper, the rigid body dynamics are implemented considering its exact geometric properties using the LGVI method. As a result, the classical model predictive control is generalized to the LGVI model of the system. The proposed method of applying convex optimization for solving MPC problems using geometric considerations is applied to a 3D pendulum, which is a rigid body supported at a frictionless pivot acting under the influence of uniform gravity with substantial invariant properties [13]. The novelty of this paper is using the LMI approach for solving the MPC control of the 3D pendulum with a variational model. This paper is organized as follows. Section II is devoted to the problem statement. Firstly, the dynamics of a 3D pendulum using LGVI are expressed, and the linearized state-space model of 3D pendulum dynamics is extracted using infinitesimal variations of parameters evolving on a manifold. Then, based on this linearized model of the system, a quadratic objective function is introduced. Section III extends the standard MPC problem on Euclidean state space to smooth manifold based on convex optimization using LMI. Simulation results are presented in section IV to prove the efficiency of using LMI in solving MPC algorithms on smooth manifolds. A comparison with the non-LMI method is also mentioned in this section. Finally, concluding remarks are presented in section V.

2 Problem Statement

2.1 3D Pendulum Dynamics

The configuration space in a 3D pendulum is a SO(3) manifold. Geometric forms of Hamilton's equations of a 3D pendulum on the configuration manifold SO(3) using LGVI method have been expressed in [4] as discrete-time forced Hamilton's equation as follows

$$\hat{\Pi}_k = \frac{1}{h}(F_k J_d - J_d F_k^T), \tag{1}$$

$$\Pi_{k+1} = F_k^T \Pi_k + h\mathcal{M}_{k+1} + hBu_{k+1}, \tag{2}$$

$$R_{k+1} = R_k F_k, \tag{3}$$

where $R_k \in \mathrm{SO}(3)$ is a rotation matrix from the body-fixed frame to the initial frame denotes the attitude of the rigid body at time k, $\Pi_k \in \mathbf{R}^3$ is the angular momentum of the pendulum expressed in the body-fixed frame, $F_k \in \mathrm{SO}(3)$ is a one-step change in R_k, J_d is a non-standard moment of inertia matrix and $J_d = \frac{1}{2}trace(J)I - J$ where J is the standard inertia matrix. $''h''$

is the time step for the discrete system. In a 3D pendulum $\mathcal{M}_k = mg\rho \times R_k^T e_3$. The hat map denotes $\hat{\cdot} : \mathbf{R} \to \mathfrak{so}(3)$ that for a given vector $\omega = \begin{pmatrix} \omega_1, \omega_2, \omega_3 \end{pmatrix}^T$ represents the skew-symmetric matrix

$$\hat{\omega} = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}.$$

As in Eucliedaian spaces, a linear vector field can estimate the Hamiltonian vector field on TSO(3) locally in an open subset of TSO(3), which is the tangent space to SO(3). Using local coordinates in the neighboring of the equilibrium point is a method of linearizing the vector field. To extract the linearized discrete Hamilton's equation for 3D pendulum as discussed in [14], a local exponential coordinate is introduced as local coordinates. The variation of the rotational matrix R is an ϵ-parameterized differentiable curve $R_{k,\epsilon}$ that takes value in SO(3), is given by [4]

$$R_{k,\epsilon} = R_k\, exp(\epsilon\, \hat{\eta}_k). \tag{4}$$

The variation of matrix R is expressed as an exponential map that returns the variations across the rotational axis η with angle ϵ. $\eta(t)$ is a differentiable curve that has value on Lie groups of rotational matrices and is identity in t_0 and t_f. The exp map is a local diffeomorphism between Lie algebra and Lie group. Other parameters' variations are also formulated as

$$F_{k,\epsilon} = F_k\, exp(\epsilon\, \hat{\xi}_k),$$

$$\Pi_{k,\epsilon} = \Pi_k + \epsilon\, \delta\Pi_k,$$

where $\delta\Pi_k$ is an infinitesimal variation of Π_k. Infinitesimal variations of the motion can be shown to be

$$\delta R_k = \frac{d}{d\epsilon}\Big|_{\epsilon=0} R_{k,\epsilon} = R_k\, \hat{\eta}_k,$$

$$\delta F_k = \frac{d}{d\epsilon}\Big|_{\epsilon=0} F_{k,\epsilon} = F_k\, \hat{\xi}_k,$$

$$\delta\Pi_k = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \Pi_{k,\epsilon} = \delta\Pi_k.$$

The infinitesimal variation of δR_{k+1} can be expressed from two points of view. On the one hand, the variation of R_{k+1} is calculated as infinitesimal variations of its parameters R_k, F_k as follows

$$\delta R_{k+1} = \delta R_k F_k + R_k \delta F_k = R_k\, \hat{\eta}_k\, F_k + R_k F_k \hat{\xi}_k\,, \tag{5}$$

on the other hand, its infinitesimal variation is calculated directly as

$$\begin{aligned} \delta R_{k+1} &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} R_{k+1,\epsilon} = \frac{d}{d\epsilon}\Big|_{\epsilon=0} R_{k+1}\, exp(\epsilon\, \hat{\eta}_{k+1}) \\ &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} R_k F_k\, exp(\epsilon\, \hat{\eta}_{k+1}) = R_k F_k \hat{\eta}_{k+1}\,. \end{aligned} \tag{6}$$

Then, the last parts of both equations (5),(6) can be equated, and the parameter η_{k+1} can be extracted as

$$\eta_{k+1} = F_k^T \eta_k + \xi_k. \tag{7}$$

Note: The relation $R^T \hat{x} R = \widehat{R^T x}$ is used. In fact, (7) is the constrained variation of the equation $R_{k+1} = R_k F_k$. Since the linearized system should be in terms of the state variables $\left[\ \eta_k,\ \delta\Pi_k\ \right]^T$, ξ_k should be replaced in terms of η_k and $\delta\Pi_k$. Due to this reason, ξ_k is calculated from (1). Firstly, the variations of (1) are obtained as follows, since $\delta\Pi_k, \delta F_k$ are not independent

$$\begin{aligned}
\delta\hat{\Pi}_k &= \frac{1}{h}(\delta F_k J_d - J_d \delta F_k^T) \\
&= \frac{1}{h}(F_k \hat{\xi}_k J_d + J_d \hat{\xi}_k F_k^T) \\
&= \frac{1}{h}(\widehat{F_k \xi_k} F_k J_d + J_d F_k^T \widehat{F_k \xi_k}).
\end{aligned}$$

Note: $\hat{x}A + A^T \hat{x} = \widehat{(trace[A] I_{3\times 3} - A)x}$

$$\begin{aligned}
\delta\hat{\Pi}_k &= \frac{1}{h}\widehat{((trace(F_k J_d) I_{3\times3} - F_k J_d) F_k \xi_k)}, \\
\delta\Pi_k &= \frac{1}{h}(trace(F_k J_d) I_{3\times3} - F_k J_d) F_k \xi_k.
\end{aligned}$$

As a result,

$$\xi_k = \beta_k \delta\Pi_k, \tag{8}$$

where

$$\beta_k = h F_k^T (trace(F_k J_d) I_{3\times3} - F_k J_d)^{-1} \in \mathbf{R}^{3\times3}.$$

By replacing (8) in (7) η_{k+1} is extrapolated as

$$\eta_{k+1} = F_k^T \eta_k + \beta_k \delta\Pi_k, \tag{9}$$

which gives the linearized rotation matrix R in terms of its rotation axis η. The dynamical equation (2) is linearized by substituting the variations of M_k. Since the torque M_k is related to the attitude of a rigid body, its variation δM_k is written as a variation of the rotational matrix

$$\delta M_k = \mathcal{M}_k \eta_k,$$

while $M_k \in \mathbf{R}^{3\times3}$ is expressed as the attitude of the rigid body and is attained by the potential field. Using (8), (9), variations of M_{k+1} is equal to

$$\delta M_{k+1} = \mathcal{M}_{k+1} \eta_{k+1} = \mathcal{M}_{k+1} F_k^T \eta_k + \mathcal{M}_{k+1} \beta_k \delta\Pi_k.$$

As a result,

$$\begin{aligned}
\delta\Pi_{k+1} =& \delta F_k^T \Pi_k + F_k^T \delta\Pi_k + h\delta\mathcal{M}_{k+1} + hB\delta u_{k+1} \\
=& -\xi_k^T F_k^T \Pi_k + F_k^T \delta\Pi_k + h\mathcal{M}_{k+1} F_k^T \eta_k \\
&+ h\mathcal{M}_{k+1} \beta_k \delta\Pi_k + hB\delta u_k \\
=& -(\beta_k \delta\Pi_k\hat{)} F_k^T \Pi_k + F_k^T \delta\Pi_k + h\mathcal{M}_{k+1} F_k^T \eta_k \\
&+ h\mathcal{M}_{k+1} \beta_k \delta\Pi_k + hB\delta u_k,
\end{aligned}$$

$$\begin{aligned}\delta\Pi_{k+1} =&((F_k^T\Pi_k)\hat{}\beta_k + F_k^T + h\mathcal{M}_{k+1}\beta_k)\delta\Pi_k \\ &+ h\mathcal{M}_{k+1}F_k^T\eta_k + hB\delta u_k.\end{aligned} \tag{10}$$

Consequently, the linearized system (1-3) is summarized as follows

$$\begin{pmatrix}\eta_{k+1}\\ \delta\Pi_{k+1}\end{pmatrix} = \begin{pmatrix}\mathcal{A}_k & \mathcal{B}_k\\ \mathcal{C}_k & \mathcal{D}_k\end{pmatrix}\begin{pmatrix}\eta_k\\ \delta\Pi_k\end{pmatrix} + \begin{pmatrix}0\\ hB\end{pmatrix}\delta u_k, \tag{11}$$

while,

$$\begin{aligned}\mathcal{A}_k &= F_k^T,\\ \mathcal{B}_k &= \beta_k,\\ \mathcal{C}_k &= h\mathcal{M}_{k+1}F_k^T,\\ \mathcal{D}_k &= F_k^T + (F_k^T\Pi_k)\hat{}\mathcal{B}_k + h\mathcal{M}_{k+1}\mathcal{B}.\end{aligned}$$

and, in a 3D pendulum

$$\begin{aligned}\delta M_k &= \mathcal{M}_k\eta_k\\ \mathcal{M}_k &= mg\rho\times R_k^T e_3 = mg\hat{\rho}(R_k e_3)\hat{}.\end{aligned}$$

2.2 Optimization Problem

As in MPC problems, a convex optimization should be solved, a quadratic cost function for the linearized system (11) is introduced as follows to minimize cost as well as energy [15, 16, 17]

$$\mathcal{J} = \mathcal{F}(R_N, \hat{\Pi}_N) + \sum_{i=0}^{N}\mathcal{L}(R_k, \hat{\Pi}_k, \hat{u}_k)$$

such that

$$\begin{aligned}\mathcal{F} &= trace(P_1(I_{3\times 3} - R_N)) + trace(P_2\hat{\Pi}_N)\\ &= \frac{1}{2}\|P_1^{1/2}(I_{3\times 3} - R_N)\|_F^2 + \frac{1}{2}\|P_2^{1/2}\hat{\Pi}_N\|_F^2,\end{aligned}$$

$$\begin{aligned}\mathcal{L}(R_k, \pi_k, u_k) &= tr(Q_1(I - R_k)) + \frac{1}{h^2}trace(Q_2(I - F_k))\\ &+ trace(u_k^T W_1 u_k)\\ &= \frac{1}{2}\|Q_1^{1/2}(I - R_k)\|_F^2 + \frac{1}{2h^2}\|Q_2^{1/2}(I - F_k)\|_F^2\\ &+ \frac{1}{2}\|W_1^{1/2}u_k\|_F^2.\end{aligned}$$

Since $\hat{\eta_k} = \log R_k$; besides, in the neighborhood of (I,I) it is true that $\delta\Pi_k \approx \Pi_k$ [17]:

$$trace(Q_1(I - R_k)) = \frac{1}{2}\|Q_1^{1/2}(I - R_k)\|_F^2$$

$$= trace(Q_1(I - exp(\hat{\eta}_k)))$$
$$, = \frac{1}{2}\eta_k^T \tilde{Q}_1 \eta_k$$

$$\begin{aligned} trace(Q_2(I - F_k)) &= \frac{1}{2h^2}\|Q_2^{1/2}(I - F_k)\|_F^2 \\ &= \frac{1}{2h^2}\|Q_2^{1/2} h(J^{-1}\delta\Pi_k\hat{)}\|_F^2 \\ &= \frac{1}{2}\delta\Pi_k^T J^{-1}\tilde{Q}_{22} J^{-1}\delta\Pi_k, \end{aligned}$$

where $\tilde{Q}_{1,22} = trace(Q_{1,22})I_{3\times3} - Q_{1,22}$ for symmetric positive definite $Q_{1,2} \in \mathbf{R}^{3\times3}$, and $\tilde{Q}_2 = J^{-1}\tilde{Q}_{22}J^{-1}$. These matrices are evaluated from the property discussed in [15], which implies that for any positive semi-definite symmetric matrix B and $c \in \mathbf{R}^3$, $\frac{1}{2}\|B^{\frac{1}{2}}\hat{c}\|_F^2 = \frac{1}{2}c^T\tilde{B}c$, where $\tilde{B} = tr(B)I_{3\times3} - B$. As a result,

$$\begin{aligned} \acute{\mathcal{L}}(\eta_k, \delta\Pi_k, \tau_k) &= \frac{1}{2}\eta_k^T\tilde{Q}_1\eta_k \\ &\quad + \frac{1}{2}\delta\Pi_k^T\tilde{Q}_2\delta\Pi_k + \frac{1}{2}\tau_k^T W\tau_k, \\ \acute{\mathcal{F}}(\eta_N, \delta\Pi_N) &= \frac{1}{2}\eta_N^T\tilde{P}_1\eta_N + \frac{1}{2}\delta\Pi_N^T\tilde{P}_2\delta\Pi_N, \end{aligned}$$

where $W = trace(W_1)I - W_1$, and $u_k \in \mathfrak{so}(3)^*$ is expressed in terms of the applied torque τ_k as $u_k = \hat{\tau}_k$ where $\mathfrak{so}(3)^*$ is the dual of $\mathfrak{so}(3)$. We rewrite the linearized system and cost function as follows

$$\zeta_{k+1} = \bar{A}\zeta_k + \bar{B}\delta u_k, \tag{12}$$

$$\mathcal{J}_N = \frac{1}{2}\zeta_N^T P\zeta_N + \frac{1}{2}\sum_{i=1}^{N}\zeta_i^T Q\zeta_i + \tau_i^T W\tau_i, \tag{13}$$

where

$$\zeta_k = \begin{pmatrix} \eta_k \\ \delta\Pi_k \end{pmatrix}, \bar{A} = \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{pmatrix}, \bar{B} = \begin{pmatrix} 0 \\ hB \end{pmatrix}$$
$$Q = \begin{pmatrix} \tilde{Q}_1 & 0 \\ 0 & \tilde{Q}_2 \end{pmatrix}, P = \begin{pmatrix} \tilde{P}_1 & 0 \\ 0 & \tilde{P}_2 \end{pmatrix}.$$

3 LMI Based Model Predictive Control

No control action is applied to the system after the instant $k+m-i$; namely, $u(k+i|k) = 0$ for $i \geq m$. From the viewpoint of the receding horizon, only the first calculated control is implemented to the system. In the next sampling time, the optimization problem $min\ \mathcal{J}$ is solved using the new measurement of the system. As a result, both m and p go one step ahead. Considering system (8), the minimization problem of the cost function is replaced by a worst-case minimization problem in each sampling time, namely the following $min-max$ problem

$$\min_{u(k+i|k)} \max_{[\mathcal{A}(k)\mathcal{B}(k)]\in\Omega} \mathcal{J}(k) \tag{14}$$

$$\mathcal{J}_N(k) = \frac{1}{2}\zeta_N^T P\zeta_N + \frac{1}{2}\sum_{i=1}^{N}\zeta(k+i|k)^T Q\zeta(k+i|k) + \tau(k+i|k)^T W\tau(k+i|k). \tag{15}$$

Maximization is on the set $\Omega = Convex Hull[\bar{A}\ \bar{B}]$, arising from stability considerations, which gives an upper bound for the Lyapunov function and leads to a robust performance objective. Then, this bound should be minimized by the use of the state feedback control law $u(k+i|k) = K\zeta(k+i|k)\, i \geq 0$. The quadratic Lyapunov function is introduced in the form of

$$V(\zeta(k|k)) = \zeta(k|k)^T P\zeta(k|k), \quad P > 0. \tag{16}$$

According to the Lyapunov stability theorem, variations of $V(\zeta(k|k))$ should be negative in order to guarantee the stability of the system. Suppose that for any $\zeta(k+i|k), u(k+i|k)$ and $i \leq 0$, the variation of Lyapunov function is smaller than a negative quadratic function, which considers being the summand of cost function such as[12]

$$\begin{aligned}\Delta V(\zeta(k+i)) &= V(\zeta(k+i+1|k)) - V(\zeta(k+i|k))\\ &= \zeta(k+i+1|k)^T P_{k+1}\zeta(k+i+1|k) - \zeta(k+i|k)^T P_k\zeta(k+i|k)\\ &\leq -(\zeta(k+i|k)^T Q_k\zeta(k+i|k) + \tau(k+i|k)^T W\tau(k+i|k)).\end{aligned} \tag{17}$$

Let us calculate the sum of both sides of (17). Firstly, the sum of the first side of it is calculated as follows [18]

$$\begin{aligned}&\sum_{i=0}^{N-1}\Delta\Big[\zeta(k+i+1|k)^T P_{k+1}\zeta(k+i+1|k)\Big]\\ &= \sum_{i=0}^{N-1}\Big[\zeta(k+i+1|k)^T P_{k+1}\zeta(k+i+1|k)\ - \zeta(k+i|k)^T P_k\zeta(k+i|k)\Big]\\ &= \zeta^T(k+N|k)P_{k+1}\zeta(k+N|k) - \zeta^T(k|k)P_k\zeta(k|k).\end{aligned}$$

Note: $\zeta(k+N|k)$ is summerized as ζ_N. Finally, it gives

$$-\zeta(k|k)^T P_k\zeta(k|k) \leq -\zeta_N^T P\zeta_N - \sum_{i=0}^{N-1}\Big[\zeta(k+i|k)^T Q_k\zeta(k+i|k)\ + \tau(k+i|k)^T W\tau(k+i|k)\Big],$$

as a result

$$\begin{aligned}-V(\zeta(k|k)) &\leq -\mathcal{J}_N(k)\\ \mathcal{J}_N(k) &\leq \zeta(k|k)^T P_k\zeta(k|k).\end{aligned} \tag{18}$$

So (18) is an upper bound for the cost function $\mathcal{J}$; namely, the problem of $max\mathcal{J}$ gives the upper bound $V(\zeta(k|k))$ for $\mathcal{J}$. It is clear that the minimization problem has been changed to determining the state feedback control gain K of $\tau(k+i|k) = K\zeta(k+i|k),\ i \geq 0$ in each sampling time k for the minimization of this upper bound of $V(\zeta(k|k))$. It means we should minimize the upper bound of $\mathcal{J}$, which is the function V. Similar to the problem of standard MPC, firstly, the first calculated input $\tau(k|k) = K\zeta(k|k)$ should be implemented to the system. In the next sampling time, the state ζ_{k+1} is measured, and optimization is repeated to compute F again. Since the linearized system has been embed on Euclidean space using the exponential map, it is a convex optimization problem and can be solved under linear matrix inequality conditions. The following theorem expresses LMI conditions on which the gain of controller K is going to calculate

Theorem 1. Let $\zeta_k = \zeta(k|k)$ be the state of the system (12) measured at the sampling time k. There are no constraints on inputs and outputs of the system. Then, the feedback matrix K in control law $\tau(k+i|k) = K\zeta(k+i|k), i \leq 0$ which minimizes the upper bound $V(\zeta(k|k))$ on the robust performance of cost function in sampling time k can be computed as follows[12]

$$K = LE^{-1}, \tag{19}$$

where the matrices $E > 0, L$ (if it exists) are obtained from the following linear minimization problem:

$$\min_{\gamma,E,L} \quad \gamma \tag{20}$$

$$subject.to.:$$

$$\begin{pmatrix} 1 & \zeta(k|k)^T \\ \zeta(k|k) & E \end{pmatrix} \geq 0, \tag{21}$$

$$\begin{pmatrix} E & EA^T + L^T B^T & EQ^{1/2} & L^T W_1^{1/2} \\ AE + BL & E & 0 & 0 \\ Q^{1/2}E & 0 & \gamma I & 0 \\ W_1^{1/2}E & 0 & 0 & \gamma I \end{pmatrix}. \geq 0 \tag{22}$$

Proof: See Appendix. A in [12]. □

4 Simulation Results

In this section, a numerical simulation is presented in order to analyze the effectiveness of the proposed method. Standard inertial matrix is chosen as $J = diag(1, 2.8, 2)$. The discretization time-step parameter $h = 0.2$. The initial angular velocity in the body-fixed frame is considered as $\Omega_0 = [0, 0, 1]$ while $\Pi = J.\Omega$, and $\eta_0 = [0, 0, 1.5]$. Solving LMI, the control gain K is computed in each iteration with control horizon 2, and only the first parameter of K is implemented to the system. The simulation results are depicted on figures (1,2). Figures show that only 7 seconds takes for the pendulum to reach its equilibrium. Simulation repeated for more complicated initial conditions, which expressed complicated starting point of the pendulum, result with initial conditions as $\Omega = [0, 1, 1]$ and $\eta_0 = [-0.5, 0, 1.5]$ is illustrated on figure (3). Using the LMI method for solving the MPC problem on the manifold is compared with the standard

MPC method on a manifold without using the LMI. Results are depicted on figures (4,5), which demonstrate fewer control efforts. This is a rest-to-rest initial condition simulation.

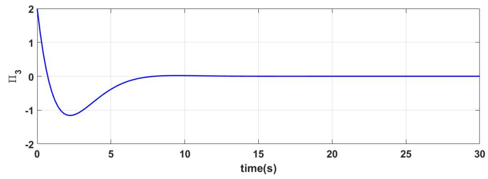

Figure 1: Angular Momentum Π_3.

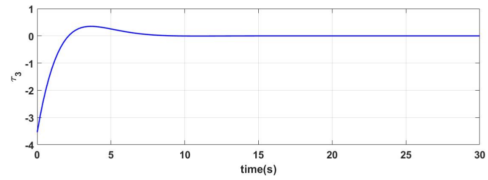

Figure 2: Input torque τ_3.

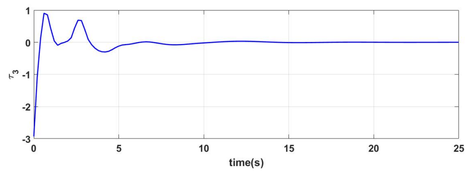

Figure 3: Input torque τ_3.

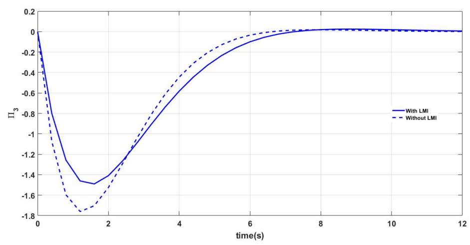

Figure 4: Comparing MPC based LMI on manifold method with regular MPC on manifold method for the parameter Π_3.

5 Conclusions

This paper formulated a model predictive method for the 3D pendulum, in which its configuration space is expressed as a manifold. Its dynamics are used as LGVI equations, and a

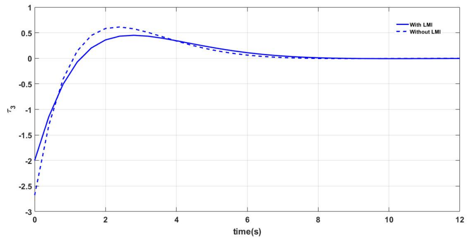

Figure 5: Comparing MPC based LMI on manifold method with regular MPC on manifold method for input torque parameter τ_3.

linearization method on manifolds has been used in order to generalize the conventional MPC methods from Euclidean spaces to manifolds. Solving MPC and calculating control gain is achieved using LMI conditions.

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طراحی کنترل پیش‌بین مدل برای کنترل پاندول سه بعدی روی منیفلد SO(۳) با استفاده از بهینه‌سازی محدب

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چکیده

روش‌های کنترل پیش‌بین بر پایه مدل (MPC) غالباً به سیستم‌های با دینامیک گسسته که در فضای برداری هموار $\mathbf{R}^n$ قرار دارند اعمال می‌شود. در حالی‌که فضای پیکره‌بندی بسیاری از سیستم‌های مکانیکی نمی‌تواند به صورت فضای اقلیدسی سراسری بیان شود. به همین جهت، روش MPC در این مقاله روی منیفلد هموار به عنوان فضای پیکره‌بندی برای کنترل وضعیت پاندول سه بعدی اعمال می‌شود. از انتگرال وردشی گروه لی (LGVI) معادلات حرکت پاندول سه بعدی به عنوان معادلات زمان گسسته استفاده می‌شود، زیرا معادلات LGVI ساختار گروه را حفظ می‌کنند. معادلات LGVI پاندول سه بعدی حول نقطه تعادل با استفاده از دیفئومورفیسم روی $\mathbf{R}^n$ خطی‌شده، سپس الگوریتم MPC به معادلات خطی‌شده اعمال می‌گردد. همچنین، مشابه روش‌های معمول MPC، مسئله بهینه‌سازی محدب حل شده و قانون کنترل در هر تکرار بدست می‌آید. در این مقاله، از نامساوی ماتریس خطی برای حل مسئله بهینه‌سازی محدب استفاده می‌شود. از یک مثال عددی برای نشان دادن عملکرد روش طراحی استفاده شده است.

کلمات کلیدی

کنترل پیش‌بین مدل، بهینه‌سازی محدب، نامساوی ماتریسی خطی، انتگرال وردشی گروه لی.