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Research Article

Necessary Optimality Conditions for Non-smooth Continuous-Time Problems Using Convexificators

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Abstract. In this paper, we develop general necessary optimality conditions of the KKT types for non-smooth continuous-time optimization problems with inequality constraints. The primary instrument in our study is the concept of a convexificator. Based on this concept, non-smooth versions of the Mangasarian-Fromovitz constraint qualification are presented. Then, we derive optimality conditions for this problem under weak assumptions. Indeed, the constraint functions and the objective function that exist in this problem are not necessarily differentiable or convex.

Keywords. Continuous-time problems, Optimality conditions, Upper semi-regular convexificator, Non-smooth analysis.

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1 Introduction

In the years that have passed, several classes of vector and scalar Continuous-time programming problems have been the subject of countless researches. First, Bellman [5] studied this class of problems in a linear case. He formulated the dual problem of this class and provided the duality relations. Investigation of the optimality conditions of KKT type for nonlinear continuous-time problems was first studied by Hanson and Mon [14]. In a smooth case, further generalizations of the FJ and KKT type of the optimality conditions for continuous-time nonlinear problems are found in [1, 12, 14, 24]. On the other hand, the FJ and KKT optimality conditions for Lipschitz continuous-time programming problems were established by Brandao et al. in [6], for the non-smooth case. Later on, by using the Clarke sub-differentials, Nobakhtian et al. in [19, 20] provided the necessary optimality conditions for non-smooth continuous-time problems with vector-valued objective functions.

In recent years, the idea of convexificator which is seen as a generalization of the sub-differential has been studied by many researchers to extend and reinforce different results in non-smooth analysis and optimization. In particular, the well-known sub-differentials, such as Michel-Penot sub-differential, Clarke sub-differential, and Treiman sub-differential, are convexificators for the locally Lipschitz functions. Unlike, well-known sub-differentials which are compact and convex, the convexificators are not necessarily convex or compact, but are always closed sets. This concept was first defined by Demyanov [8] in the form of a convex and compact set, as a generalization of the idea of lower concave and upper convex approximations. Afterward, Jeyakumar and Luc in [16] used a closed and nonconvex set instead of a convex and compact set and they introduced what is now known as the nonconvex convexifiector.

Many researchers have obtained different results concerning FJ and KKT type necessary optimality conditions for non-smooth optimization problems with using convexificators (see [3, 4, 8, 9, 10, 11, 13, 16, 22, 23]). The upper semi-regular convexificator defined in [11] is a strengthened version of an upper convexifactor. Further, for the locally Lipschitz function, one can have the upper semi-regular convexificators smaller than the Michel-Penot sub-differential, Clarke sub-differential, Mordukhovich sub-differential, and Treiman sub-differential, so the optimality conditions obtained in terms of upper semi-regular convexificators are sharper. Our purpose in this paper is to introduce a new generalization of the Mangasarian-Fromovitz constraint qualification and following that to obtain an actual result of necessary optimality conditions in the form of KKT type for non-smooth continuous-time problems with inequality constraints with using convexificator. As mentioned above, the convexificator is viewed as a generalization of the sub-differential, so it leads to more robust results in optimization. Indeed, the results in this perusal are correct with the convexificator replaced by known sub-differentials such as Clarke and Mordukhovich sub-differentials where the functions are locally Lipschitz. The main result of this paper might be used to generalize the duality and optimality conditions for non-smooth continuous-time optimization problems in future research.

The structure of the next sections of this work is as follows. In the second section, we introduce the notations and give the basic definitions of convexificators, support functions, integration of Multifunctions, and derive some preparative results from being used in the residual of this paper. In the third section, we introduce a new constraint qualification for non-smooth continuous-time problems with inequality constraints via convexificator and we establish KKT type necessary optimality conditions.

2 Notations and Preliminaries

Let $\mathcal{Z} \subset L_m^\infty[0,\mathcal{T}]$ be a nonempty, open and convex where $(L_m^\infty[0,\mathcal{T}], \|.\|_\infty)$ is the Banach space of all m-dimensional vector-valued Lebesgue measurable essentially bounded functions defined on the $[0,\mathcal{T}] \subset \mathbb{R}$, with the essential norm $\|.\|_\infty$ appointed by

$$\|z\|_\infty = \max_{1\le j\le m} \operatorname{ess\,sup}\{|z_j(t)|, \quad t \in [0,\mathcal{T}]\},$$

for $z(t) = (z_1(t), \cdots, z_m(t)) \in \mathbb{R}^m$. We consider the following non-smooth continuous-time problem:

$$\begin{aligned} (CTP) \qquad \min \quad & \Theta(\mathcal{R}) = \int_0^{\mathcal{T}} \theta(t,\mathcal{R}(t))dt, \\ s.t. \quad & \rho_i(t,\mathcal{R}(t)) \le 0, \quad i \in \mathcal{I} = \{1,\ldots,r\}, \quad a.e. \quad t \in [0,\mathcal{T}], \\ & \mathcal{R} \in \mathcal{Z}, \end{aligned}$$

where the functions $t \to \theta(t,\mathcal{R}(t))$ and $t \to \rho_i(t,\mathcal{R}(t))$ are integrable and Lebesgue measurable for all $\mathcal{R} \in \mathcal{Z}$. We define $\rho_i(t,\mathcal{R}(t)) = \Phi_i(\mathcal{R})(t)$, $i \in \mathcal{I}$ and $\theta(t,\mathcal{R}(t)) = \Psi(\mathcal{R})(t)$, where $\Phi_i(\cdot)$ and $\Psi(\cdot)$ are maps from $\mathcal{Z}$ into the normed space $\Lambda_1^1[0,\mathcal{T}]$ of all functions which are Lebesgue measurable and essentially bounded defined on the compact interval $[0,\mathcal{T}]$. In the rest of this paper, the set of feasible solutions of the (CTP) is denoted as Ω. In other words,

$$\Omega = \{\mathcal{R} \in \mathcal{Z} : \ \rho_i(t,\mathcal{R}(t)) \le 0 \quad a.e. \ t \in [0,\mathcal{T}], \quad i \in \mathcal{I}\}.$$

Next, we reminisce some concepts and substantial structures of non-smooth analysis that are needed in the following of this paper. Most of the concepts and substance included here can be found in [7] where the reader can find rather references, details and discussions.

Suppose that $\psi : Y \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be an extended real valued function defined on the Banach space Y. The upper and lower Dini directional derivatives of ψ at $\bar{y}$ in direction $d \in Y$ are introduced respectively by

$$\psi^+(\bar{y}; d) := \limsup_{t \downarrow 0} \frac{\psi(\bar{y} + td) - \psi(\bar{y})}{t},$$
$$\psi^-(\bar{y}; d) := \liminf_{t \downarrow 0} \frac{\psi(\bar{y} + td) - \psi(\bar{y})}{t}.$$

Whenever for function ψ at $\bar{y}$ in direction $d \in Y$, the upper and lower Dini directional derivatives are equal, we say that the function ψ has directional derivative at $\bar{y}$ in direction $d \in Y$ and is denoted as $\psi'(\bar{y}; d)$. If ψ is Fréchet differentiable at $\bar{y}$ with Fréchet derivative $\nabla\psi(\bar{y})$, then for all $d \in \mathbb{R}^m$, $\psi'(\bar{y}; d) = \langle \nabla\psi(\bar{y}), d \rangle$. Also, it is worth mentioning that both the upper and lower Dini derivatives definitely exist for locally Lipschitz functions.

Now, assume $\overline{\mathcal{R}} \in \mathcal{Z}$ and $d \in L_m^\infty[0, \mathcal{T}]$. We have

$$\rho_i^+(t, \overline{\mathcal{R}}(t); d(t)) := \Phi_i^+(\overline{\mathcal{R}}; d)(t) := \limsup_{\lambda \downarrow 0} \frac{\Phi_i(\overline{\mathcal{R}} + \lambda d)(t) - \Phi_i(\overline{\mathcal{R}})(t)}{\lambda},$$

a.e. $t \in [0, T]$. According to the above hypotheses, it follows that for every $\overline{\mathcal{R}} \in \mathcal{Z}$ and $d(t) \in L_m^\infty[0, \mathcal{T}]$ the functions

$$t \to \psi^+(t, \overline{\mathcal{R}}(t); d(t))$$
$$t \to \rho_i^+(t, \overline{\mathcal{R}}(t); d(t)), \quad i \in \mathcal{I},$$

are integrable and Lebesgue measurable.

Now, to achieve the desired results, we recall the definition of the integration of multi-functions. Let $\Gamma : [0, \mathcal{T}] \to \mathbb{R}^m$ be a multi-function defined on $[0, \mathcal{T}]$. The integral of Γ, which denoted by $\int_0^{\mathcal{T}} \Gamma(t)dt$, is defined by the following subset of $\mathbb{R}^m$,

$$\int_0^{\mathcal{T}} \Gamma(t)dt := \left\{ \int_0^{\mathcal{T}} \gamma(t)dt : \quad \gamma \in \mathcal{S}(\Gamma) \right\},$$

where $\mathcal{S}(\Gamma)$ is the following set,

$$\mathcal{S}(\Gamma) := \{\gamma \in L_1^m[0, \mathcal{T}],\ \gamma(t) \in \Gamma(t)\ a.e.\ t \in [0, \mathcal{T}]\},$$

where $L_1[0, \mathcal{T}] := \{\eta : [0, \mathcal{T}] \to \mathbb{R} : \|\eta\|_1 := \int_0^{\mathcal{T}} |\eta(t)| < \infty\}$.

If for measurable multi-function Γ, there exists an integrable function $\beta : [0, \mathcal{T}] \to \mathbb{R}_+$ such that

$$|\Gamma(t)| = \sup_{\xi \in \Gamma(t)} \|\xi\| \leq \beta(t), \qquad a.e.\ t \in [0, \mathcal{T}],$$

thus, we say that Γ is integrally bounded.

Suppose that for Banach space Y, the dual space of continuous linear functionals on Y equipped with the weak* topology is denoted by Y^*. In this case, the support function of a nonempty subset Δ in Y is the function $H_\Delta : Y^* \to \overline{\mathbb{R}}$, which defined as follows

$$H_\Delta(\xi) = \sup\{\langle \xi, y\rangle : y \in \Delta\},$$

where $\langle \cdot, \cdot \rangle$ is the standard duality pairing between Y^* and Y.

Next, we state some fundamental results for support functions that are required in the sequel.

Proposition 1. ([15]) Let $\mu,\ \lambda \geq 0$ be given scalars. Then, for nonempty closed convex subsets $\Sigma,\ \Delta$ of Y, we have

$$\Sigma \subseteq \Delta \quad \text{iff} \quad H_\Sigma(\xi) \leq H_\Delta(\xi), \quad \forall \xi \in Y^*,$$
$$\mu H_\Sigma(\xi) + \lambda H_\Delta(\xi) = H_{\{\mu\Sigma + \lambda\Delta\}}(\xi), \quad \forall \xi \in Y^*.$$

Theorem 1. ([2, Proposition 8.6.2]) For an integrably bounded multi-function Γ with compact values, we have

$$H_{\int_0^T \Gamma(t)dt}(v) = \int_0^T H_{\Gamma(t)}(v)dt, \qquad \forall\ v \in \mathbb{R}^n.$$

We now recall the definitions of convexificator and some of its significant properties from [16]. As mentioned before, this notion plays a key role in the main results of this paper.

- We say that the function $\psi : Y \to \overline{\mathbb{R}}$ have an upper convexificator (u.c or Jeyakumar-Luc sub-differential [23]) (or lower convexificator(l.c)) at $y \in Y$ if there is a weak* closed set $\partial^*\psi(y) \subset Y^*$ (or $\partial_*\psi(y) \subset Y^*$) so that for every $d \in Y$,

  $$\psi^-(y;d) \leq \sup_{\xi \in \partial^*\psi(y)} \langle \xi, d\rangle \Big(\text{or } \psi^+(y;d) \geq \inf_{\xi \in \partial_*\psi(y)} \langle \xi, d\rangle\Big).$$

  A closed set $\partial^*\psi(y) \subset Y^*$ is said to be a convexificator of ψ at y iff it is both l.c and u.c of ψ at y.

- The function $\psi : Y \to \overline{\mathbb{R}}$ is said to have an upper-regular convexificator(u.r.c)(or lower regular convexificator(l.r.c)) at $y \in Y$ if there is a weak* closed set $\partial^*\psi(y) \subset Y^*$ (or $\partial_*\psi(y) \subset Y^*$) so that for every $d \in Y$,

  $$\psi^+(y;d) = \sup_{\xi \in \partial^*\psi(y)} \langle \xi, d\rangle \Big(\text{or } \psi^-(y;d) = \inf_{\xi \in \partial_*\psi(y)} \langle \xi, d\rangle\Big).$$

Some significant properties of convexificators, such as neither necessarily convex nor necessarily compact, make it possible to study a large class of non-smooth problems using this concept. We note that if a continuous function $\psi : \mathbb{R}^n \to \overline{\mathbb{R}}$ embrace a locally bounded Jeyakumar-Luc sub-differential at z, then it is locally Lipschitz near this point (see [16]).

Now, we recall from [10], the definition of upper semi-regular convexificators which will be beneficial in what follows.

- We say that, the function $\psi : Y \to \overline{\mathbb{R}}$ have an upper-semi regular convexificator(u.s.r.c) at $y \in Y$ if there is a closed set $\partial^*\psi(y) \subset Y^*$ so that for every $d \in Y$,

$$\psi^+(y;d) \leq \sup_{\xi \in \partial^*\psi(y)} \langle \xi, d \rangle.$$

Obviously, an u.r.c of ψ is also an u.s.r.c of ψ and each u.s.r.c is an u.p. Moreover, convex hull of an u.s.r.c of a locally Lipschitz function may be strictly contained in Michel-Penot and Clarke sub-differential(see Example 2.1 of [16]).

Example 1. Assume that function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ is defined as follows:

$$\psi(z) := \begin{cases} \sin 2z, & \text{if } z \in [0, +\infty) \cap \mathbb{Q}, \\ z^3 - 3z, & \text{if } z \in (-\infty, 0] \cap \mathbb{Q}, \\ 0, & \text{Otherwise,} \end{cases}$$

where $\mathbb{Q}$ is the set of rational numbers. In this case, by slightly calculating, the lower and upper Dini derivatives of function ψ at the point $z = 0$, are obtained respectively by:

$$\psi^-(0;d) = 0, \quad (\forall d \in \mathbb{R}),$$

$$\psi^+(0;d) = \begin{cases} 2d, & \text{if } d \geq 0, \\ -3d, & \text{if } d < 0. \end{cases}$$

Therefore, for function ψ at $z = 0$, the nonconvex set $\{-3, 2\}$ is an u.s.r.c. Also, the convex set $[-3, 2]$ is too another u.s.r.c for this function and so they are u.c for ψ at $z = 0$. Indeed, convexificators are not necessarily unique.

Since for all $d \in Y$, $\psi^-(y;d) \leq \psi^+(y;d)$, it follows that an u.s.r.c is also an u.c of ψ at y. The following example illustrates that the reverse of this statement is not necessarily true.

Example 2. Suppose that the propound function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ is as follows.

$$\psi(z) = \begin{cases} \tan 2z, & \text{if } z \in \mathbb{Q}, \\ 0, & \text{if } z \notin \mathbb{Q}. \end{cases}$$

Then,

$$\psi^+(0;d) = 2\max\{0,d\},$$
$$\psi^-(0;d) = 2\min\{0,d\}.$$

Observe that the set $\partial^*\psi(0) = \{-1,1\}$ is an u.c of ψ at $z = 0$. On the other hand, since by considering $d = 1$, we have,

$$\sup_{\xi\in\partial^*\psi(0)} \langle \xi, 1\rangle = 1 < \psi^+(0;1) = 2.$$

The set $\partial^*\psi(0)$ is not an u.s.r.c.

Remark 1. Note that the set $\partial^*\psi(x) = \{\nabla\psi(x)\}$ is a unique u.r.c for any differentiable function such as ψ. Moreover, we know that any locally Lipschitz function is differentiable almost everywhere. Thus, these functions have a u.r.c on a dense set. Also, if ψ is locally Lipschitz, then the Mordukhovich sub-differential $\partial_M\psi(x)$ [18], Michel-Penot sub-differential $\partial^\diamond\psi(x)$ [17], Clarke sub-differential $\partial_C\psi(x)$ [7] and Treiman sub-differential $\partial_T\psi(x)$ [21] are examples of u.s.r.c for ψ.

In the following Lemma, we derive some calculus rules for upper semi-regular convexificator under convenient conditions.

Lemma 1. Let $\psi = (\psi_1,\dots,\psi_k)$ be a continuous vector function from Banach space Y to $\mathbb{R}^k$, and for each $l = 1,\dots,k$, the function ψ_l admits an u.s.r.c, $\partial^*\psi_l(\tilde{y})$ at $\tilde{y}\in Y$. Also, assume that,

$$g(y) = \max\{\psi_l(y):\ l = 1,\dots,k\},$$

and $I(\tilde{y}) = \{l:\ g(\tilde{y}) = \psi_l(\tilde{y})\}$. Then $co\{\bigcup_{l\in I(\tilde{y})}\partial^*\psi_l(\tilde{y})\}$ is an u.s.r.c of g at $\tilde{y}$(for the nonempty set S, the convex hall of S is denoted by coS).

Proof. Suppose, without loss of generality, that $I(\tilde{y}) = \{1,\dots,q\}$. Thus

$$g(\tilde{y}) = \psi_1(\tilde{y}) = \cdots = \psi_q(\tilde{y}) > \psi_j(\tilde{y}),\quad \forall j = q+1,\dots,k.$$

From the continuity of ψ_l, for every y in a neighborhood of $\tilde{y}$, we have

$$g(y) = \max\{\psi_l(y) : l = 1,\dots,q\}.$$

Therefore

$$\begin{aligned} g^+(\tilde{y};d) &= \limsup_{t\downarrow o}\frac{g(\tilde{y}+td)-g(\tilde{y})}{t}\\ &= \limsup_{t\downarrow o}\frac{\max\{\psi_1(\tilde{y}+td),\dots,\psi_q(\tilde{y}+td)\}-g(\tilde{y})}{t}\end{aligned}$$

$$
\begin{aligned}
&= \limsup_{t\downarrow o} \max\Big\{\frac{\psi_1(\tilde{y}+td)-\psi_1(\tilde{y})}{t},\dots,\frac{\psi_q(\tilde{y}+td)-\psi_q(\tilde{y})}{t}\Big\}\\
&= \max\Big\{\limsup_{t\downarrow o}\frac{\psi_1(\tilde{y}+td)-\psi_1(\tilde{y})}{t},\dots,\limsup_{t\downarrow o}\frac{\psi_q(\tilde{y}+td)-\psi_q(\tilde{y})}{t}\Big\}\\
&= \max\Big\{\psi_1^+(\tilde{y};d),\dots,\psi_q^+(\tilde{y};d)\Big\}\\
&\le \sup\Big\{\langle\eta,d\rangle : \eta\in\partial^*\psi_1(\tilde{y})\cup\ldots\cup\partial^*\psi_q(\tilde{y})\Big\}\\
&\le \sup\Big\{\langle\eta,d\rangle : \eta\in co\{\bigcup_{l=1}^{q}\partial^*\psi_l(\tilde{y})\}\Big\}
\end{aligned}
$$

Hence, $co\{\bigcup_{l\in I(\tilde{y})}\partial^*\psi_l(\tilde{y})\}$ is an u.s.r.c of g at $\tilde{y}$. □

3 Main Results

In this section, we focus on getting optimality conditions for (CTP). Indeed, we deduce the KKT type of the necessary optimality conditions for a large class of nonconvex and non-differentiable continuous-time problems. Using the idea of u.s.r.c, we generalize the Mangasarian-Fromovitz constraint qualification for a non-smooth problem where the objective and constraint functions are continuous. The Generalized Gordan Theorem [25], will help to obtain the results of this section.

Let us introduce the following non-smooth analogue of the generalized Mangasarian–Fromovitz constraint qualification named by (GMFCQ).

Definition 1. The GMFCQ is satisfied at a feasible point $\tilde{x}$ of (CTP), if there exists a non-zero vector $v \in L^\infty[0,\mathcal{T}]$ such that for all $i \in \mathcal{I}$,

$$\sup_{\zeta\in\partial^*\rho_i(t,\tilde{x}(t))}\langle\zeta, v(t)\rangle < 0, \qquad a.e.\ t\in A_i(\tilde{x}),$$

where $A_i(\tilde{x}) := \{t \in [0,\mathcal{T}] : \rho_i(t,\tilde{x}(t)) = 0\}$.

Next, we are ready to demonstrate our result of KKT type necessary conditions in terms of u.s.r.c.

Theorem 2. Let GMFCQ holds at $\bar{\mathcal{R}} \in \Omega$. Also, suppose that $\theta(t,\cdot)$ and $\rho_i(t,\cdot)$, $i \in \mathcal{I}$ are continuos functions at $\bar{\mathcal{R}}$, and admit bounded upper semi-regular convexificator $\partial^*\theta(t,\bar{\mathcal{R}})$ and $\partial^*\rho_i(t,\bar{\mathcal{R}})$ for all $i \in \mathcal{I}$. Then there exists $\overline{\lambda} \in L_r^\infty[0,\mathcal{T}]$, such that

$$0\in\int_0^{\mathcal{T}}\Big[\mathrm{co}\partial^*\theta(t,\bar{\mathcal{R}}(t))+\sum_{i=1}^{r}\overline{\lambda}_i(t)\mathrm{co}\partial^*\rho_i(t,\bar{\mathcal{R}}(t))\Big]dt, \tag{1}$$

$$\overline{\lambda}(t) \geq 0, \quad a.e. \ t \in [0, \mathcal{T}], \tag{2}$$

$$\overline{\lambda}_i(t)\rho_i(t, \bar{\mathcal{R}}(t)) = 0, \quad a.e. \ t \in [0, \mathcal{T}], \quad i \in \mathcal{I}. \tag{3}$$

Proof. We get,

$$\rho(t, \bar{\mathcal{R}}(t)) = \max_{1 \leq i \leq r} \rho_i(t, \bar{\mathcal{R}}(t)), \quad a.e. \ t \in [0, \mathcal{T}].$$

Thus, we reformulate (CTP) in the following tantamount form.

$$\begin{aligned} \min \quad & \Theta(\mathcal{R}) = \int_0^{\mathcal{T}} \theta(t, \mathcal{R}(t)) dt, \\ s.t. \quad & \rho(t, \mathcal{R}(t)) \leq 0, \quad a.e. \quad t \in [0, \mathcal{T}], \\ & \mathcal{R} \in \mathcal{Z}. \end{aligned}$$

Now, we show that the following system has no solution $d \in L_r^\infty[0, \mathcal{T}]$.

$$\Theta^+(\bar{\mathcal{R}}; d) < 0, \tag{4}$$

$$\rho^+(t, \bar{\mathcal{R}}(t); d(t)) < 0, \quad a.e. \ t \in A(\bar{\mathcal{R}}) := \{t \in [0, \mathcal{T}] : \rho(t, \bar{\mathcal{R}}(t)) = 0\}. \tag{5}$$

Assume, to the contrary, that the system (4) and (5) has a solution $\hat{d} \in L_r^\infty[0, \mathcal{T}]$. According to continuity of the functions and the limsup properties , there exists a real number $\delta > 0$ such that for every $\gamma \in (0, \delta)$,

$$\begin{aligned} \rho(t, \bar{\mathcal{R}}(t) + \gamma \hat{d}(t)) \leq 0, & \quad a.e. \ t \in [0, \mathcal{T}], \\ \Theta(\bar{\mathcal{R}} + \gamma \hat{d}(t)) < \Theta(\bar{\mathcal{R}}), & \quad \bar{\mathcal{R}} + \gamma \hat{d}(t) \in \mathcal{Z}. \end{aligned}$$

This means that $\bar{\mathcal{R}} + \gamma \hat{d}$ is a feasible solution of (CTP) for $\gamma \in (0, \delta)$. This conflicts with the fact that $\hat{z}$ is an optimal solution for problem (CTP). Hence there does not exists $d \in L_r^\infty[0, \mathcal{T}]$ which satisfy in the systems (4) and (5). Applying the Generalized Gordan Theorem yields the existence of $\lambda_0 \geq 0$, $\hat{\lambda}(t) \geq 0$ a.e. $t \in [0, \mathcal{T}]$, not all identically zero, such that

$$0 \leq \lambda_0 \Theta^+(\bar{\mathcal{R}}; d) + \int_{A(\bar{\mathcal{R}})} \hat{\lambda}(t) \rho^+(t, \bar{\mathcal{R}}(t); d(t)) dt, \quad \forall d \in L_r^\infty[0, \mathcal{T}].$$

Setting $\lambda(t) = \hat{\lambda}(t)$, if $t \in A(\bar{\mathcal{R}})$ and $\lambda(t) = 0$, otherwise, we acquire

$$\begin{aligned} 0 & \leq \lambda_0 \Theta^+(\bar{\mathcal{R}}; d) + \int_0^{\mathcal{T}} \lambda(t) \rho^+(t, \bar{\mathcal{R}}(t); d(t)) dt \\ & = \lambda_0 \limsup_{\mu \downarrow 0} \frac{\int_0^{\mathcal{T}} [\Psi(\bar{\mathcal{R}} + \mu d)(t) - \Psi(\bar{\mathcal{R}})(t)] dt}{\mu} + \int_0^{\mathcal{T}} \lambda(t) \rho^+(t, \bar{\mathcal{R}}(t); d(t)) dt \\ & \leq \int_0^{\mathcal{T}} \lambda_0 \Big[\limsup_{\mu \downarrow 0} \frac{\Psi(\bar{\mathcal{R}} + \mu d)(t) - \Psi(\bar{\mathcal{R}})(t)}{\mu} \Big] dt + \int_0^{\mathcal{T}} \lambda(t) \rho^+(t, \bar{\mathcal{R}}(t); d(t)) dt \end{aligned}$$

$$= \int_0^{\mathcal{T}} \lambda_0 \theta^+(t, \bar{\mathcal{R}}(t); d(t))dt + \int_0^{\mathcal{T}} \lambda(t)\rho^+(t, \bar{\mathcal{R}}(t); d(t))dt,$$

for all $d \in L_r^\infty[0, \mathcal{T}]$. Let $I(t, \bar{\mathcal{R}}(t)) = \{i \in \mathcal{I} : \rho(t, \bar{\mathcal{R}}) = \rho_i(t, \bar{\mathcal{R}})\}$. Similar to the proof of Lemma 1, we have

$$\rho^+(t, \bar{\mathcal{R}}(t); d(t)) = \max\{\rho_i^+(t, \bar{\mathcal{R}}(t); d(t)) : i \in I(t, \bar{\mathcal{R}}(t))\}. \tag{6}$$

Now, let $d_0 \in L_r^\infty[0, \mathcal{T}]$ be the vector which is satisfied in $(GMFCQ)$. Therefore for every $i \in \mathcal{I}$,

$$\sup_{\zeta \in \partial^* \rho_i(t, \bar{\mathcal{R}}(t))} \langle \zeta, d_0(t) \rangle < 0, \qquad a.e.\ t \in A_i(\bar{\mathcal{R}}). \tag{7}$$

According to relationship (7), since ρ_i admit an u.s.r.c, we have

$$\rho_i^+(t, \bar{\mathcal{R}}(t); d_0(t)) < 0, \quad a.e.\ t \in [0, \mathcal{T}], \quad i \in \mathcal{I}.$$

The equation in (6) in turn gives the following strict inequality

$$\rho^+(t, \bar{\mathcal{R}}(t); d_0(t)) < 0, \quad a.e.\ t \in [0, \mathcal{T}]. \tag{8}$$

Indeed, if $\lambda_0 = 0$, then we have

$$0 \le \int_0^{\mathcal{T}} \lambda(t)\rho^+(t, \bar{\mathcal{R}}(t); d(t))dt, \quad \forall d \in L_r^\infty[0, \mathcal{T}].$$

Accordingly, by the Generalized Gordan Theorem, there is no $d \in L_r^\infty[0, \mathcal{T}]$ such that

$$\rho^+(t, \bar{\mathcal{R}}(t); d(t)) < 0, \quad a.e.\ t \in [0, \mathcal{T}].$$

This is a contradiction with (8). So, $\lambda_0 \neq 0$. We put $\mu(t) = \frac{\lambda(t)}{\lambda_0}$, and hence we obtain

$$\begin{aligned}
0 &\le \int_0^{\mathcal{T}} \theta^+(t, \bar{\mathcal{R}}(t); d(t))dt + \int_0^{\mathcal{T}} \mu(t)\rho^+(t, \bar{\mathcal{R}}(t); d(t))dt \\
&\le \int_0^{\mathcal{T}} \Big[\sup_{\xi(t) \in \mathrm{co}\partial^* \theta(t, \bar{\mathcal{R}}(t))} \langle \xi(t), d(t) \rangle + \mu(t) \sup_{\eta(t) \in \mathrm{co}\partial^* \rho(t, \bar{\mathcal{R}}(t))} \langle \eta(t), d(t) \rangle \Big] dt \\
&= \int_0^{\mathcal{T}} \Big[H_{\mathrm{co}\partial^* \theta(t, \bar{\mathcal{R}}(t))}(d(t)) + \mu(t) H_{\mathrm{co}\partial^* \rho(t, \bar{\mathcal{R}}(t))}(d(t)) \Big] dt \\
&= \int_0^{\mathcal{T}} \Big[H_{\{\mathrm{co}\partial^* \theta(t, \bar{\mathcal{R}}(t)) + \mu(t)\mathrm{co}\partial^* \rho(t, \bar{\mathcal{R}}(t))\}}(d(t)) \Big] dt,
\end{aligned}$$

for every $d \in L_r^\infty[0, \mathcal{T}]$. Insomuch the above inequality holds for every $d \in L_r^\infty[0, \mathcal{T}]$, it holds, in special, for constant functions $d(t) = u \in L_r^\infty[0, \mathcal{T}]$.

Sine $\partial^*\theta(t,\bar{\mathcal{R}}(t))$ and $\partial^*\rho(t,\bar{\mathcal{R}}(t))$ are bounded upper semi-regular convexificators, it can be easily verified that the multi-function

$$t \longrightarrow \mathrm{co}\partial^*\theta(t,\bar{\mathcal{R}}(t)) + \mu(t)\mathrm{co}\partial^*\rho(t,\bar{\mathcal{R}}(t))$$

is integrably bounded and compact-valued. By Theorem 1, we have

$$\begin{aligned} H_{\{0\}}(d) = 0 \leq \int_0^{\mathcal{T}} \Big[H_{\{\mathrm{co}\partial^*\theta(t,\bar{\mathcal{R}}(t))+\mu(t)\mathrm{co}\partial^*\rho(t,\bar{\mathcal{R}}(t))\}}(d)\Big]dt \\ = H_{\{\int_0^{\mathcal{T}}[\mathrm{co}\partial^*\theta(t,\bar{\mathcal{R}}(t))+\mu(t)\mathrm{co}\partial^*\rho(t,\bar{\mathcal{R}}(t))]dt\}}(d). \end{aligned}$$

Therefore,

$$0 \in \int_0^{\mathcal{T}} [\mathrm{co}\partial^*\theta(t,\bar{\mathcal{R}}(t)) + \mu(t)\mathrm{co}\partial^*\rho(t,\bar{\mathcal{R}}(t))]dt. \tag{9}$$

It can be inferred from (9) and the definition of integration of multi-functions that there exists a measurable function $\eta(t) \in \mathrm{co}\partial^*\rho(t,\bar{\mathcal{R}}(t))$ a.e. $t \in [0,\mathcal{T}]$ such that

$$0 \in \int_0^{\mathcal{T}} [\mathrm{co}\partial^*\theta(t,\bar{\mathcal{R}}(t)) + \mu(t)\eta(t)]dt. \tag{10}$$

On the other hand,by the Lemma 1, $co\{\bigcup_{i\in I(t,\bar{\mathcal{R}})}\partial^*\rho_i(t,\bar{\mathcal{R}}(t))\}$ is an u.s.r.c for $\rho(t,\cdot)$, we have

$$\eta(t) \in co\{\bigcup_{i\in I(t,\bar{\mathcal{R}})} \partial^*\rho_i(t,\bar{\mathcal{R}}(t))\}.$$

Therefore, there exists $\kappa \in L_r^\infty[0,\mathcal{T}],\ \kappa(t) \geq 0$ a.e. $t \in [0,\mathcal{T}]$ such that

$$\eta(t) \in \sum_{i=1}^{r} \kappa_i(t)\partial^*\rho_i(t,\bar{\mathcal{R}}(t)), \qquad \sum_{i=1}^{r} \kappa_i(t) = 1.$$

By defining $\overline{\lambda}_i(t) := \mu(t)\kappa_i(t),\ i \in \mathcal{I}$, it follows from (10) the proof is complete. □

4 Conclusion

Since the Clarke and Michel-Penot subdifferentials of a locally Lipschitz function are bounded u.s.r.c, the results of Theorem 2 and Lemma 1 in this work are valid with the convexificators replaced by these subdifferentials. Also, the main result of this paper is an actual theorem that might be used to generalize the optimality conditions and duality for non-smooth continuous-time optimization problems and non-smooth multi-objective continuous-time problems in future research.

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