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Research Article

A Fuzzy Distance Measure for Fuzzy Numbers

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Abstract. A fuzzy distance measure is introduced in this paper to evaluate the fuzzy distance between two fuzzy numbers. For this purpose, α-values of fuzzy numbers are used to develop an integral-based fuzzy distance measure. The properties of the proposed fuzzy distance measure are verified. The proposed fuzzy distance measure is also compared with other fuzzy distance measures.

Keywords. Fuzzy number, α-values, Fuzzy distance measure, Robustness.

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1 Introduction

Fuzzy sets [47] have been employed in many real-life applications. In many applications often there is need a distance measure between two fuzzy numbers to solve the problems. Many distance measures have been proposed for fuzzy sets including exact [1, 6, 8, 14, 20, 21, 26, 27, 29, 31, 32, 33, 36, 40] and fuzzy quantities [3, 4, 10, 11, 18, 24, 22, 34, 37, 41]. The idea behind a fuzzy distance is that if the quantities are reported as fuzzy numbers, their distance is also expected to be a fuzzy number. Other distance measures were conducted based on other types of imprecision including intuitionistic fuzzy sets and interval-valued fuzzy numbers [15, 17, 19, 28, 39, 42, 43, 44, 45, 48, 49, 5].

A distance measure is an essential issue in many real-life applications that involve error measurements between non-exact quantities. Many studies, as mentioned before, have calculated the distance between two fuzzy numbers as exact numbers. However, an exact distance is not usually reasonable in the fuzzy domain since an exact value for distance between two fuzzy numbers may outcome in loss of essential information under imprecision. Thus, it is reasonable to say that the distance between two fuzzy numbers should be expressed as a fuzzy number, too. Furthermore, many existing fuzzy distances do not satisfy some proper properties of a distance measure expected in the fuzzy domain. In this paper, a new fuzzy distance measure was proposed for fuzzy numbers. It was shown that the proposed fuzzy distance measure satisfied all properties for an absolute error distance in the fuzzy domain. The main advantages of the proposed fuzzy distance measure over others were also extensively illustrated.

This paper is organized as follows: Section 2 reviews some concepts of fuzzy numbers. Section 3 introduces fuzzy distance measures between two fuzzy numbers. The main properties of the proposed fuzzy distance measures are also discussed and compared with other fuzzy distances. The main contributions of this study will be discussed in Section 4.

2 Fuzzy Numbers

This section briefly reviews some concepts and terminology related to α-values of fuzzy numbers used throughout this paper.

Let $\mathbb{X}$ be a universal set. A fuzzy set of $\widetilde{A}$ is defined by its membership function $\widetilde{A} : \mathbb{X} \to [0,1]$. The set $\widetilde{A}[\alpha] := \{x \in \mathbb{X} : \widetilde{A}(x) \geq \alpha\}$ is called the α-cut of $\widetilde{A}$ [30]. $\widetilde{A}$ is called a fuzzy number (**FN**) on $\mathbb{X} = \mathbb{R}$ if

1. There exists a unique $x^*_{\widetilde{A}} \in \mathbb{R}$ with $\widetilde{A}(x^*_{\widetilde{A}}) = 1$, and
2. the set $\widetilde{A}[\alpha] = \{x \in \mathbb{R} : \widetilde{A}(x) \geq \alpha\}$ is a non-empty nested closed interval in $\mathbb{R}$, for every $\alpha \in (0,1]$.

Such interval is presented by $\widetilde{A}[\alpha] = [\widetilde{A}^L_\alpha, \widetilde{A}^U_\alpha]$ in which $\widetilde{A}^L_\alpha = \inf\{x : x \in \widetilde{A}[\alpha]\}$ and $\widetilde{A}^U_\alpha = \sup\{x : x \in \widetilde{A}[\alpha]\}$. Moreover, a fuzzy number of $\widetilde{A}$ is an LR-fuzzy number (**LRFN**) if there exist real numbers of a, l_a and r_a with $l_a, r_a \geq 0$, and strictly decreasing and continuous functions of $L, R : [0,1] \to [0,1]$ such that

$$\widetilde{A}(x) = \begin{cases} L(\frac{a-x}{l_a}), & a - l_a \leq x \leq a, \\ R(\frac{x-a}{r_a}), & a < x \leq a + r_a, \\ 0, & x \in \mathbb{R} - [a-l, a+r]. \end{cases} \tag{1}$$

where $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$. An LR-fuzzy number of $\widetilde{A}$ can be simply denoted by $(a; l_a, r_a)_{LR}$. The most commonly used LR-fuzzy numbers are triangular fuzzy numbers (**TFN**s) in which the shape functions of L and R are given by $L(x) = R(x) = 1 - x$, for all $x \in [0,1]$. The membership function of a **TFN** of $\widetilde{A} = (a; l, r)_T$ is denoted by

$$\widetilde{A}(x) = \begin{cases} \frac{x-a+l_a}{l_a}, & a - l_a \leq x \leq a, \\ \frac{a+r_a-x}{r_a}, & a \leq x \leq a + r_a, \\ 0, & x \in \mathbb{R} - [a-l_a, a+r_a]. \end{cases} \tag{2}$$

Some common operations between two LR-fuzzy numbers of $\widetilde{A} = (a; l_a, r_a)_{LR}$ and $\widetilde{B} = (b; l_b, r_b)_{LR}$ can be defined as follows [30]:

1) (Addition) $\widetilde{A} \oplus \widetilde{B} = (a + b; l_a + l_b, r_a + r_b)_{LR}$.

3) (Scalar multiplication):

$$\lambda \otimes \widetilde{A} = \begin{cases} (\lambda a; \lambda l_a, \lambda r_a)_{LR}, & if\ \lambda > 0, \\ (\lambda a; -\lambda r_a, -\lambda l_a)_{RL}, & if\ \lambda < 0. \end{cases} \tag{3}$$

Here, the notion of α-values of **FN**s is recalled.

Definition 1. [23] The α-values of a **FN** $\widetilde{A}$ is a mapping $\widetilde{A}_\alpha : [0,1] \to \mathbb{R}$ defined by:

$$\widetilde{A}_\alpha = \begin{cases} \widetilde{A}^L_{2\alpha}, & \alpha \in [0, 0.5], \\ \widetilde{A}^U_{2(1-\alpha)}, & \alpha \in [0.5, 1], \end{cases} \tag{4}$$

where $\widetilde{A}^L_\alpha$, $\widetilde{A}^U_\alpha$ show the lower and upper limits of $\widetilde{A}[\alpha]$, respectively.

Example 1. Let $\widetilde{A} = (a; l_a, r_a)_{LR}$ be an LR-**FN**. From Definition 1, one finds that:

$$\widetilde{A}_\alpha = \begin{cases} a - l_a L^{-1}(2\alpha), & 0 \leq \alpha \leq 0.5, \\ a + r_a R^{-1}(2(1-\alpha)), & 0.5 \leq \alpha \leq 1. \end{cases}$$

For instance,

1. If $\widetilde{A} = (a; l_a, r_a)_T$ is a **TFN**, then,

$$\widetilde{A}_\alpha = \begin{cases} (a - l_a) + 2 l_a \alpha, & 0 \leq \alpha \leq 0.5, \\ a + r_a - 2 r_a (1-\alpha), & 0.5 \leq \alpha \leq 1. \end{cases}$$

2. Let $\widetilde{A} = (a; l_a, r_a)_{LR}$ with $L(x) = \sqrt{1-x^3}$ and $R(x) = 1 - x^5$ then:

$$\widetilde{A}_\alpha = \begin{cases} a - l_a \sqrt[3]{1-4\alpha^2}, & 0 \le \alpha \le 0.5, \\ a + l_a \sqrt[5]{2\alpha - 1}, & 0.5 \le \alpha \le 1. \end{cases}$$

The relationship between α-values and α-cuts of **FN**s can be investigated by the following lemma [23].

Lemma 1. Let $\{\widetilde{A}_\alpha\}_{\alpha\in[0,1]}$, called α-values, be a strictly decreasing function of α and $\widetilde{A}_{0.5}$ be a constant number. Then, $\{\widetilde{A}_\alpha\}_{\alpha\in[0,1]}$ can construct a **FN** $\widetilde{A}$ whose α-cuts is $\widetilde{A}[\alpha] = [\widetilde{A}_{1-\alpha/2}, \widetilde{A}_{\alpha/2}]$. In addition, if $\widetilde{A}$ is a **FN**, then its α-values are given by:

$$\widetilde{A}_\alpha = \begin{cases} \widetilde{A}^L_{2\alpha}, & \alpha \in [0, 0.5], \\ \widetilde{A}^U_{2(1-\alpha)}, & \alpha \in (0.5, 1]. \end{cases} \tag{5}$$

Proof. Given the α-values of $\{\widetilde{A}_\alpha\}_{\alpha\in[0,1]}$, it is easy to verify that $[\widetilde{A}_{1-\alpha_2/2}, \widetilde{A}_{\alpha_2/2}] \subseteq [\widetilde{A}_{1-\alpha_1/2}, \widetilde{A}_{\alpha_1/2}]$. Therefore, from Representation Theorem [30], the α-cuts of $\widetilde{A}[\alpha] = [\widetilde{A}_{1-\alpha/2}, \widetilde{A}_{\alpha/2}]$ can construct the fuzzy number of $\widetilde{A}$. Therefore, $\widetilde{A}$ is a **FN**. Now, let $\widetilde{A}$ be a **FN** with the following α-values:

$$\widetilde{A}_\alpha = \begin{cases} \widetilde{A}^U_{2\alpha}, & \alpha \in [0, 0.5], \\ \widetilde{A}^L_{2(1-\alpha)}, & \alpha \in (0.5, 1]. \end{cases} \tag{6}$$

Then, it is readily seen that

1) $\widetilde{A}_\alpha$ is a strictly decreasing function with relative to $\alpha \in [0,1]$ and $\widetilde{A}_{0.5}$ is a constant number,

2) $\widetilde{A}[\alpha] = [\widetilde{A}_{1-\alpha/2}, \widetilde{A}_{\alpha/2}]$,

which completes the proof. □

Definition 2. [25] Let $\widetilde{A}$ and $\widetilde{B}$ be two **FN**s. It is said that $\widetilde{A} \le \widetilde{B}$, if $\widetilde{A}_\alpha \le \widetilde{B}_\alpha$ for any $\alpha \in [0,1]$.

We will use such ordering to define a fuzzy distance measure in the next section.

Remark 1. [25] It is notable that the addition and scalar multiplication of **FN**s (mentioned before) can be reevaluated based on their α-values as follows:

$$(\widetilde{A} \oplus \widetilde{B})_\alpha = \widetilde{A}_\alpha + \widetilde{B}_\alpha,$$

$$(\lambda \otimes \widetilde{A})_\alpha = \begin{cases} \lambda \widetilde{A}_\alpha, & \text{if} \quad \lambda > 0, \\ \lambda \widetilde{A}_{1-\alpha}, & \text{if} \quad \lambda < 0. \end{cases}$$

3 Fuzzy Distance Measure

In this section, a fuzzy distance measure between two **FN**s is introduced according to a two-step procedure. In the first step, an exact distance measure is proposed. Using such distance measure, a fuzzy distance measure is proposed. To do these, a popular definition of a (non-fuzzy) distance measure between two **FN**s is first recalled.

Definition 3. A mapping $d : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to [0,\infty)$ is called a (non-fuzzy) distance measure if any $\widetilde{A}, \widetilde{B}$ and $\widetilde{C} \in \mathcal{F}(\mathbb{R})$ meets the following conditions:

1) $d(\widetilde{A},\widetilde{B}) = 0$ if and only if $\widetilde{A} = \widetilde{B}$,

2) $d(\widetilde{A},\widetilde{B}) = d(\widetilde{B},\widetilde{A})$,

3) $d(\widetilde{A},\widetilde{C}) \leq d(\widetilde{A},\widetilde{B}) + d(\widetilde{B},\widetilde{C})$.

Theorem 1. For two **FN**s of $\widetilde{A}$ and $\widetilde{B}$, define

$$d_a(\widetilde{A},\widetilde{B}) = \int_0^1 g_{\widetilde{A},\widetilde{B}}(\alpha)d\alpha,$$

where

1)

$$g_{\widetilde{A},\widetilde{B}}(\alpha) = \int_{\alpha/2}^{1-\alpha/2} |\widetilde{A}_\beta - \widetilde{B}_\beta| f_{(\alpha/2,0.5,1-\alpha/2)}(\beta)d\beta,$$

2) $f_{\alpha/2,0.5,1-\alpha/2}$ denotes the triangular density function on $[\alpha/2, 1-\alpha/2]$ defined as

$$f_{(\alpha/2,0.5,1-\alpha/2)}(\beta) = \begin{cases} \frac{4(\beta-\alpha/2)}{(1-\alpha)^2}, & \alpha/2 \leq \beta \leq 0.5, \\ \frac{4(1-\alpha/2-\beta)}{(1-\alpha)^2}, & 0.5 \leq \beta \leq 1-\alpha/2. \end{cases} \tag{7}$$

Then, $d_a : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to [0,\infty)$ is a non-fuzzy distance measure.

Proof. To prove, it is enough to show that d_a satisfies assertions (1)-(3) in Definition 4. To check assertion (1), note that $d_a(\widetilde{A},\widetilde{B}) = 0$ if and only if $g_{\widetilde{A},\widetilde{B}} = 0$ for any $\alpha \in [0,1]$, i.e. $\widetilde{A}_\beta = \widetilde{B}_\beta$ for any $\beta \in [0,1]$ which means $\widetilde{A} = \widetilde{B}$. Assertion (2) is also immediately followed. To prove assertion (3), by the conventional triangular inequality, we have

$$|\widetilde{A}_\beta - \widetilde{C}_\beta| \leq |\widetilde{A}_\beta - \widetilde{B}_\beta| + |\widetilde{B}_\beta - \widetilde{C}_\beta|,$$

for any $\beta \in [0,1]$. Integrating on both sides with respect to $f_{(\alpha/2,0.5,1-\alpha/2)}(\beta)$ on $[\alpha/2, 1-\alpha/2]$, will result in $g_{\widetilde{A},\widetilde{C}}(\alpha) \leq g_{\widetilde{A},\widetilde{B}}(\alpha) + g_{\widetilde{B},\widetilde{C}}(\alpha)$ for any $\alpha \in [0,1]$. This means $\widetilde{d_a}(\widetilde{A},\widetilde{C}) \leq (\widetilde{d_a}(\widetilde{A},\widetilde{B}) + \widetilde{d_a}(\widetilde{B},\widetilde{C}))$ which completes the proof. □

It is noticeable that $f_{(\alpha/2,0.5,1-\alpha/2)}(\beta)$ cares about points near 0.5 (that is those values near to modal of fuzzy numbers) more than other values in $[\alpha/2,1-\alpha/2]$. Interestingly, d_a can be interpreted as the double expected value of uniform distribution on $[0,1]$ and a triangular density function on $[\alpha/2,1-\alpha/2]\subseteq[0,1]$.

Now, a procedure is suggested to extend the non-fuzzy distance measure of d_a as "about d_a". First, a common notion of fuzzy distance measure is reviewed.

Definition 4. [24] We say that $\widetilde{D}:\mathcal{F}(\mathbb{R})\times\mathcal{F}(\mathbb{R})\to\mathcal{F}([0,\infty))$ is a fuzzy distance measure (**FDM**) if any $\widetilde{A},\widetilde{B}$ and $\widetilde{C}\in\mathcal{F}(\mathbb{R})$ meet the following conditions:

1) $\widetilde{D}(\widetilde{A},\widetilde{B})=I\{0\}$ if and only if $\widetilde{A}=\widetilde{B}$,

2) $\widetilde{D}(\widetilde{A},\widetilde{B})=\widetilde{D}(\widetilde{B},\widetilde{A})$,

3) $\widetilde{D}(\widetilde{A},\widetilde{C})\preceq(\widetilde{D}(\widetilde{A},\widetilde{B})\oplus\widetilde{D}(\widetilde{B},\widetilde{C}))$.

Lemma 2. For two **FN**s of $\widetilde{A}$ and $\widetilde{B}$, define

$$(\widetilde{d_a}(\widetilde{A},\widetilde{B}))_\alpha=\begin{cases}\int_0^1 g_{\widetilde{A},\widetilde{B}}(2w\alpha)dw, & 0\le\alpha\le 0.5,\\ \int_0^1 g_{\widetilde{A},\widetilde{B}}(1-2(1-w)(1-\alpha))dw, & 0.5\le\alpha\le 1.\end{cases}\tag{8}$$

Then, $\widetilde{d_a}$ is a **FN**.

Proof. For every $\alpha_1<\alpha_2$, first, note that $I_{\alpha_2}=[\alpha_2/2,1-\alpha_2/2]\subseteq I_{\alpha_1}=[\alpha_1/2,1-\alpha_1/2]$ and $f_{(\alpha_2/2,0.5,1-\alpha_2/2)}(\beta)\le f_{(\alpha_1/2,0.5,1-\alpha_1/2)}(\beta)$ for any $\beta\in[0,1]$. This simply implies that $g_{\widetilde{A},\widetilde{B}}(\alpha_2)\le g_{\widetilde{A},\widetilde{B}}(\alpha_1)$ for every $0\le\alpha_1<\alpha_2\le 1$, i.e. $g_{\widetilde{A},\widetilde{B}}(\alpha)$ is a decreasing function on $[0,1]$. This immediately concludes that $(\widetilde{d_a}(\widetilde{A},\widetilde{B}))_\alpha$ is also a decreasing function with respect to α. Moreover, it is seen that $(\widetilde{d_a}(\widetilde{A},\widetilde{B}))_{0.5}=d_a(\widetilde{A},\widetilde{B})$ is a constant number. These verify that $\widetilde{d_a}$ is a **FN**. □

Theorem 2. Recall assumptions in Lemma 2. Then $\widetilde{d_a}:\mathcal{F}(\mathbb{R})\times\mathcal{F}(\mathbb{R})\to\mathcal{F}([0,\infty))$ is a **FDM**.

Proof. We show that $\widetilde{d_a}(\widetilde{A},\widetilde{B})$ meets all the conditions of Definition 4. Future, $\widetilde{d_a}(\widetilde{A},\widetilde{B})=I\{0\}$ if and only if for any $\alpha\in[0,1]$, $(\widetilde{d_a}(\widetilde{A},\widetilde{B}))_\alpha=0$. This simply concludes that $\widetilde{A}_\alpha=\widetilde{B}_\alpha$ for any $\alpha\in[0,1]$ or $\widetilde{A}=\widetilde{B}$. This shows (1).The assertion (2) can be immediately followed. To prove assertion (3), it is enough to see that $g_{\widetilde{A},\widetilde{C}}(\alpha)\le g_{\widetilde{A},\widetilde{B}}(\alpha)+g_{\widetilde{B},\widetilde{C}}(\alpha)$. This simply implies that $(\widetilde{d_a}(\widetilde{A},\widetilde{C}))_\alpha\le((\widetilde{d_a}(\widetilde{A},\widetilde{B}))_\alpha+(\widetilde{d_a}(\widetilde{B},\widetilde{C})))_\alpha$ for any $\alpha\in[0,1]$ which completes the proof by Remark 1 and Definition 2. □

Next, the most important properties needed for an absolute error distance are verified in a fuzzy domain.

Lemma 3. If $\widetilde{A},\widetilde{B},\widetilde{C}\in\mathcal{F}(\mathbb{R})$ and $\lambda\in\mathbb{R}$, then

1) $\widetilde{d_a}(\widetilde{A}\oplus\widetilde{C},\widetilde{B}\oplus\widetilde{C})=\widetilde{d_a}(\widetilde{A},\widetilde{B})$.

2) $\widetilde{d_a}(\lambda\otimes\widetilde{A},\lambda\otimes\widetilde{B})=|\lambda|\otimes\widetilde{d_a}(\widetilde{A},\widetilde{B})$.

Proof. For three **FN**s of $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$, it is easy to verify that

1) $g_{\widetilde{A}\oplus\widetilde{C},\widetilde{B}\oplus\widetilde{C}}(\alpha) = g_{\widetilde{A},\widetilde{B}}(\alpha)$,
2) $g_{\lambda\otimes\widetilde{A},\lambda\otimes\widetilde{B}}(\alpha) = |\lambda| g_{\widetilde{A},\widetilde{B}}(\alpha)$,

for any $\alpha \in [0,1]$. Therefore,

$$(\widetilde{d_a}(\widetilde{A}\oplus\widetilde{C},\widetilde{B}\oplus\widetilde{C}))_\alpha = \begin{cases} \int_0^1 g_{\widetilde{A}\oplus\widetilde{C},\widetilde{B}\oplus\widetilde{C}}(2w\alpha)dw & 0 \le \alpha \le 0.5, \\ \int_0^1 g_{\widetilde{A}\oplus\widetilde{C},\widetilde{B}\oplus\widetilde{C}}(1-\alpha))dw & 0.5 \le \alpha \le 1, \end{cases} \tag{9}$$

$$= \begin{cases} \int_0^1 g_{\widetilde{A},\widetilde{B}}(2w\alpha)dw & 0 \le \alpha \le 0.5, \\ \int_0^1 g_{\widetilde{A},\widetilde{B}}(1-\alpha))dw & 0.5 \le \alpha \le 1, \end{cases} = (\widetilde{d_a}(\widetilde{A},\widetilde{B}))_\alpha, \tag{10}$$

for any $\alpha \in [0,1]$. This concludes that $\widetilde{d_a}(\widetilde{A}\oplus\widetilde{C},\widetilde{B}\oplus\widetilde{C}) = \widetilde{d_a}(\widetilde{A},\widetilde{B})$. The second assertion can be also verified due to $g_{\lambda\otimes\widetilde{A},\lambda\otimes\widetilde{B}}(\alpha) = |\lambda| g_{\widetilde{A},\widetilde{B}}(\alpha)$ for any $\alpha \in [0,1]$. □

Remark 2. If two fuzzy numbers of $\widetilde{A}$ and $\widetilde{B}$ reduce to non-fuzzy quantities of a and b, then it can be checked that $\widetilde{d_a}(\widetilde{A},\widetilde{B}) = I(|a-b|)$ which is the conventional absolute error distance.

Note that a distance measure should be able to tolerate small errors in evaluating the membership functions that is the distance between two **FN**s should not change if the variation of the membership functions is sufficiently small [**?**]. Here, a notion of robustness for a **FDM** is defined.

Definition 5. Let $\widetilde{D} : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to \mathcal{F}([0,\infty))$ be a **FDM**. We say $\widetilde{D}$ is robust, if for any given pair of **FN**s $(\widetilde{A},\widetilde{B})$ and a sequence of pair **FN**s of $\{(\widetilde{A}_n,\widetilde{B}_n)\}$ with $d_H(\widetilde{A}_n,\widetilde{A}) \to 0$ and $d_H(\widetilde{B}_n,\widetilde{B}) \to 0$ as $n \to \infty$ we have $d_H(\widetilde{D}(\widetilde{A}_n,\widetilde{B}_n),\widetilde{D}(\widetilde{A},\widetilde{B})) \to 0$ as $n \to \infty$ in which d_H is the Hausdorff distance measure between two **FN**s defined by $d_H(\widetilde{A},\widetilde{B}) = \sup_{\alpha\in[0,1]} |\widetilde{A}_\alpha - \widetilde{B}_\alpha|$.

Now, the robustness of the proposed **FDM** of $\widetilde{d_a}$ is examined by the following theorem.

Theorem 3. The **FDM** of $\widetilde{d_a}$ introduced in Lemma 2 is robust.

Proof. For a given pair of **FN**s $(\widetilde{A},\widetilde{B})$ and a sequence of **FN** pairs $\{(\widetilde{A}_n,\widetilde{B}_n)\}$, assume that $d_H(\widetilde{A}_n,\widetilde{A}) \to 0$ and $d_H(\widetilde{B}_n,\widetilde{B}) \to 0$ as $n \to \infty$. First, for every $\alpha \in [0,1]$, note that

$$(\widetilde{d_a}(\widetilde{A},\widetilde{B}))_\alpha \le \sup_{\alpha\in[0,1]} |\widetilde{A}_\alpha - \widetilde{B}_\alpha| = d_H(\widetilde{A},\widetilde{B}). \tag{11}$$

By Theorem 2 and Equation (11), it follows that

$$\begin{aligned}(\widetilde{d_a}(\widetilde{A}_n,\widetilde{B}_n))_\alpha &\le (\widetilde{d_a}(\widetilde{A}_n,\widetilde{A}))_\alpha + (\widetilde{d_a}(\widetilde{B}_n,\widetilde{A}))_\alpha, \\ &\le (\widetilde{d_a}(\widetilde{A}_n,\widetilde{A}))_\alpha + (\widetilde{d_a}(\widetilde{B}_n,\widetilde{B}))_\alpha + (\widetilde{d_a}(\widetilde{A},\widetilde{B}))_\alpha, \\ &\le d_H(\widetilde{A}_n,\widetilde{A}) + d_H(\widetilde{B}_n,\widetilde{B}) + (\widetilde{d_a}(\widetilde{A},\widetilde{B}))^\beta_\alpha.\end{aligned}$$

Similarly, we have $(\widetilde{d_a}(\widetilde{A},\widetilde{B}))_\alpha \le d_H(\widetilde{A}_n,\widetilde{A}) + d_H(\widetilde{B}_n,\widetilde{B}) + (\widetilde{d_a}(\widetilde{A}_n,\widetilde{B}_n))_\alpha$. This concludes that $|(\widetilde{d_a}(\widetilde{A}_n,\widetilde{B}_n))_\alpha - (\widetilde{d_a}(\widetilde{A},\widetilde{B}))_\alpha| \le d_H(\widetilde{A}_n,\widetilde{A}) + d_H(\widetilde{B}_n,\widetilde{B})$ for every $\alpha \in [0,1]$. Therefore, $d_H(\widetilde{d_a}(\widetilde{A}_n,\widetilde{B}_n),\widetilde{d_a}(\widetilde{A},\widetilde{B})) \to 0$ as $n \to \infty$. This shows that $\widetilde{d_a}$ is robust. □

Example 2. Consider two **TFN**s $\widetilde{A}_1 = (7; 1.3, 0.7)_T$ and $\widetilde{A}_2 = (3; 0.4, 0.2)_T$. First, note that $d_a(\widetilde{A}_1, \widetilde{A}_2) = 3.898$. Furthermore, the **FDM** of $\widetilde{d}_a$ between $\widetilde{A}_1$ and $\widetilde{A}_2$ was evaluated as "about 3.898" whose membership degree is depicted in Figure 1.

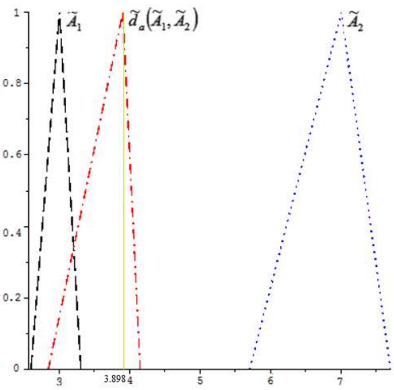

Figure 1: Plot of $\widetilde{d}_a(\widetilde{A}_1, \widetilde{A}_2)$ in Example 2.

Remark 3. Chakraborty and Chakraborty [10] suggested a fuzzy distance measure ($\widetilde{d}^*$) for generalized fuzzy numbers. Ganbari and Nuraei [16] verified that Chakraborty and Chakraborty's distance is not always a generalized triangular fuzzy number. Further, it may be a nonnegative generalized triangular fuzzy number of $\widetilde{d}^*$. Additionally, $\widetilde{d}^*(\widetilde{A} \oplus \widetilde{C}, \widetilde{B} \oplus \widetilde{C}) \neq \widetilde{d}^*(\widetilde{A}, \widetilde{B})$ which is true for $\widetilde{D}_a$ as discussed in Lemma 3. Voxman [41] proposed a notion of fuzzy absolute error distance of $\widetilde{\Delta}$ between two fuzzy numbers. However, the main disadvantage of such distance measure is that $\widetilde{\Delta}(\widetilde{A} \oplus \widetilde{C}, \widetilde{B} \oplus \widetilde{C}) \neq \widetilde{\Delta}(\widetilde{A}, \widetilde{B})$ which is expected for an absolute error distance in a fuzzy environment. Abbasbandy and Hajighasemi [4], introducing a definition for fuzzy distance measure, proposed a symmetric triangular fuzzy distance measure $\widetilde{d}$ which satisfies all conditions in Lemma 3. But, the main shortcoming of $\widetilde{d}$ is that if $\widetilde{d}(\widetilde{A}, \widetilde{B}) = I\{0\}$ then it is not necessary to have $\widetilde{A} = \widetilde{B}$. Sadi-Nezhad et al. [35] proposed a triangular fuzzy absolute error distance measure of $\widetilde{d}$ between two triangular fuzzy numbers of $\widetilde{A} = (a; l_a, r_a)_T$ and $\widetilde{B} = (b; l_b, r_b)_T$. Therefore, the possible applications of a such distance measure are limited in space of **FN**s. Another disadvantage of $\widetilde{d}$ is that $\widetilde{d}(\widetilde{A}, \widetilde{A}) \neq (0; 0, 0)_T$. Beigi et al. [7] proposed a triangular absolute error fuzzy distance measure of $\widetilde{d}$ for two triangular fuzzy numbers of $\widetilde{A} = (a; l_a, r_a)_T$ and $\widetilde{B} = (b; l_b, r_b)_T$. However, it can be observed that $\widetilde{d}(\lambda \otimes \widetilde{A}, \lambda \otimes \widetilde{B}) \neq |\lambda| \otimes \widetilde{d}(\widetilde{A}, \widetilde{B})$ in the case where $\lambda < 0$. Moreover, such a fuzzy distance measure can be only used for triangular **FN**s. Chen and Wang [13] proposed a probability fuzzy absolute error distance measure [12] between two LR-fuzzy numbers of $\widetilde{A}$ and $\widetilde{B}$. However, in cases where $\widetilde{d}(\widetilde{A}, \widetilde{B}) = I\{0\}$, one can observe that $\widetilde{A} = \widetilde{B}$ is not guaranteed. Moreover, Hesamian and Akbari [24] proposed a fuzzy distance measure between two LR-**FN**s based on the absolute value of fuzzy numbers and an extended subtraction operation. The potential advantages of such fuzzy distance lie in its robustness and satisfaction of all conditions in Lemma 3. However, the proposed fuzzy absolute error distance can be used only for LR-**FN**s while the proposed method is applicable on any type of **FN**s. Moreover, a simpler procedure to compute the fuzzy distance between two **FN**s is suggested in this paper compared to Hesamian and Akbari's method.

4 Conclusion

This paper proposed a fuzzy distance measure between two fuzzy numbers using α-values as a fuzzy number. The main expected properties of the proposed fuzzy distance measure were also verified in the space of fuzzy numbers. Calculating the fuzzy distance between two fuzzy numbers was demonstrated using a numerical example. The advantages of the proposed fuzzy distance measure were also examined and compared with other existing ones. Future studies should be devoted to extending a notion of distance measure in the space of intuitionistic fuzzy numbers or type-2 fuzzy numbers.

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