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Research Article

Numerical Solution of Vasicek Equation by Using Brownian Wavelets and Multiple Ito-Integral

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Abstract. In this paper, we present a new approach to solving stochastic differential equations and the Vasicek equation by using Brownian wavelets and multiple Ito-integral. Firstly, the calculation of the multiple Ito-integral based on the structure of Brownian motion is presented and the error of Ito-integrate computation is minimized under this condition. Then, the Brownian wavelets 1D and 3D based on coefficients Brownian motion are introduced. After that, a system of linear and nonlinear equations of coefficients Brownian motion is obtained such that by solving this system the approximate solution of the Vasicek equation is obtained. In the last section, some numerical examples are given.

Keywords. Stochastic differential equation, Vasicek equation, Brownian motion, Brownian wavelets, Ito-integral.

MSC. 90C34; 90C40.

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1 Introduction

For the First time, microscopic Brownian motion (abbreviated to BM) was discovered by Robert Brown a scientist who observed through a microscope the random swarming motion of pollen grains in water. The theory of Brownian motion was developed by Bachelier [3], also by Einstein in his paper [4], which used Brownian motion to estimate Avogadro's number. Now, BM also called the Wiener process has been defined by Wiener in 1923 [14]. Wiener proved that there exists a special case of Brownian motion with continuous paths in general form; Brownian motion is a Gaussian process and a Markov process. So, it is in the theory of stochastic process. Also history of Brownian motion and related processes we cite Meyer, Kahane [6], [7], and Yor [11]. Firstly, Vasicek was accounting for a bond pricing model that interest rate. The short rate dynamics is defined as a diffusion process with constant parameters [10]. The bond price is based on some assumptions; it has the feature that on a given date, the market price of risk. Vasicek's model is a special case of the Ornstein-Uhlenbeck (O-U) process, with constant volatility. This implies that the short rate is both a Gaussian process and a Markovian process. Vasicek's model also exhibits mean-reversion and is able to capture monetary authority's behavior of setting target rates. Also, the historical experience of interest rates justifies the Ornstein-Uhlenbeck specification. So, the pedagogical value of this model in stochastic term structure modeling is considerable. Also, there are many problems that can be modeled as stochastic differential equation (SDE). In this study, we present a new numerical method for solving stochastic differential equations. Firstly, we introduce some definition and numerical approximation of Ito-integral based on Brownian motion after that, the Brownian wavelets 1D and 3D-Brownian motion based on coefficients Brownian motion are introduced. By using this result, a system of linear and nonlinear equations of coefficients Brownian motion is obtained such that by solving this system the approximation solution of the Vasicek equation is optioned.

2 Basic Definitions and Theorems

The Karhunen-Loeve expansion operates much the same as Fourier Expansion, but on the space $L^2(\omega, F, P)$ of equivalence classes of random variables is a Hilbert space, and the inner product $E[X_1X_2]$, also inducted norm $X_2 = \sqrt{E(X^2)}$ and random functions X_t that is continuous-parameter, $t \in [a,b]$ considered over $[a,b]$. These random functions are found by study the Hilbert-Schmidt integral operator, which has the covariance of the random function as a kernel. This expansion decomposes the stochastic process by projecting every variable onto an orthonormal basis for space spanned by the operator's Eigen-functions, which equivalence of the positive eigenvalues thereof. The coefficients of the infinite linear combination are expected to be random variables, hence, for all $t \in [a,b]$, they are the projection of a random variable onto a deterministic orthogonal basis. As a result, these random coefficients are also orthogonal in $L^2(\omega, F, P)$, namely, they are uncorrelated.

Theorem 1 (The Karhunen-Loeve Expansion)**.** Let X_t be a zero-mean square-integrable stochastic process defined over a probability space (ω, F, P) and indexed over a closed and bounded interval $[a, b]$, with continuous covariance function $k(s, t)$. Then $k(s, t)$ is a Mercer kernel and letting e_k be an orthonormal basis on $L^2([a, b])$ formed by the eigen functions of T_{kx} with respective eigenvalues λ_k, X_t admits the following representation

$$x_t = \sum_{i=1}^{\infty} z_i e_i(t) \quad in \quad L^2(\omega),$$

where the convergence is in L^2 uniform in t and $Z_i(\omega) = \int_a^b x_t(\omega) e_i(t) dt$.

Furthermore, the random variables Z_k have zero-mean, are uncorrelated and have variance λ_k, i.e., we have:

- $E(z_i) = 0$.
- $E(z_i z_j) = \delta_{ij}\lambda_j$ where δ_{ij} is Kronecker delta.
- $Var(z_i) = \lambda_j$.

Note that by generalizations of Mercer's theorem we can replace the interval $[a, b]$ with other compact spaces C and the Lebesgue measure on $[a, b]$ with a Borel measure whose support is C.

Proof. See [5]. □

3 Brownian Wavelet

In this section, we introduce the Brownian wavelet and some another definition. In the first (see [12]), we fix the time interval $[t_0, t_1]$. Therefore, the standard Brownian motion has a Karhunen–Loeve series expansion of the form such that the $z_i \sim N(0, 1),\ i = 1, 2, \cdots$ are independent random variables and ψ_i are orthonormal bases of the Hilbert space with inner product as:

$$< f, g >= \int_{t_0}^{t_1} f(\tau) g(\tau) d\tau.$$

The Gaussian random variables are the projections of the Brownian motion onto the basic functions

$$z_i = \int_{t_0}^{t} \psi_i(\tau) d\beta(\tau).$$

The series expansion can be interpreted as the following representation for the differential of Brownian motion

$$d\beta(t) = \sum_{i=0}^{\infty} z_i \psi_t dt.$$

We have the following theorem.

Theorem 2. Stochastic processes $\beta(t)$ that has series expansion of this form, is a Brownian motion.

$$\beta(t,\omega)=\sum_{n=0}^{\infty}a_n\psi_n(t)G_n(\omega). \tag{1}$$

Proof. See [12]. □

As we know, the basic functions are introduced in various forms and here we use the following form:

$$\beta(t)=\sum_{i=0}^{\infty}z_i\int_{t_0}^{t}\psi_i(\tau)d\beta(\tau),$$

where the functions $\psi_i(t),\ i=1,2,3,\cdots$, are introduced in the following:

Definition 1. (Piecewise function): We define the $\psi_i(t)$ as follow:

$$\psi_1(t)=\begin{cases}2t & 0\le t<1/2,\\ 2-2t & 1/2\le t<1,\\ 0 & \text{o.w.}\end{cases}$$

$$\psi_n=\psi_1(2^jt-k),$$

where $t\in[0,1]$ and for $n>1,\ n=2^j+k$. Some of $\psi_n(t)$ heve been shown in Figures 1 and 2.

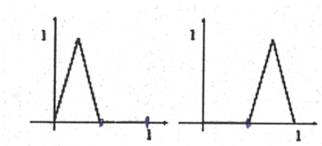

Figure 1: $\psi_2(t),\psi_3(t)$.

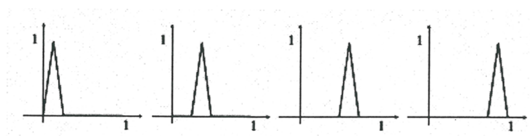

Figure 2: $\psi_4(t),\psi_5(t),\psi_6(t),\psi_7(t)$.

Definition 2. (Sinus function): We consider a well-defined

$$\psi_n(t)=\sqrt{2}\sin[(n-1/2)\pi t]. \tag{2}$$

Some other $\psi_n(t)$ heve been shown in Figures 3 and 4, where $1\le n$.

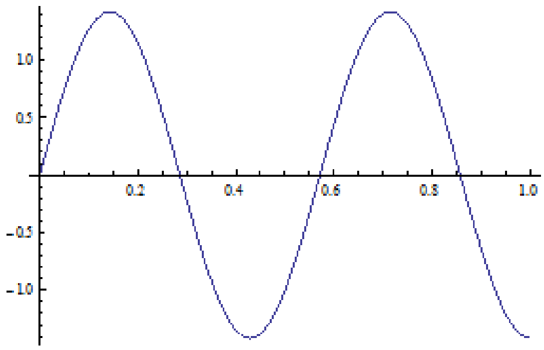

Figure 3: $\psi_3(t) = \sqrt{2}\sin[(5/2)\pi t]$.

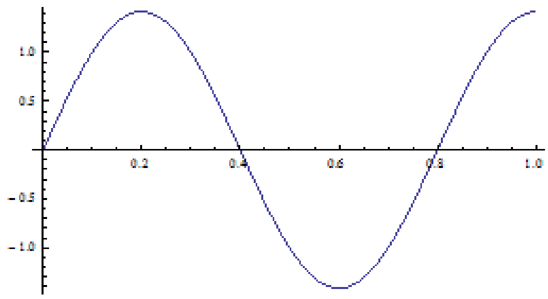

Figure 4: $\psi_4(t) = \sqrt{2}\sin[(7/2)\pi t]$.

3.1 1D and 3D- Brownian Wavelet

The one-dimensional Brownian wavelets can be constructed as basic sinus functions $\psi_n(t)$, such that are orthonormal, so we have the following equations:

$$< \psi_n, \psi_{n-1} >= 0,$$

$$< \psi_n, \psi_n >= 1.$$

In [12], it has been represented that the wavelet coefficients are

$$a_n = \frac{1}{(n-1/2)\pi} \qquad n = 1, 2, \cdots,. \tag{3}$$

The three-dimensional wavelets [10] can be constructed as separable products of 1D-wavelets by applying 1D-wavelets in three dimensional and three directions (x, y, z). The volume $F(x, y, z)$ at first is filtered the x-direction, resulting in a high-pass image $H(x, y, z)$ and a low-pass image $L(x, y, z)$. Both of the low-pass and high-pass are then filtered along the y-direction, resulting in four decomposed sub-volumes: Low-Low, Low-High, High-Low and High-High. Then each of these four sub-volumes are filtered along the z-direction, resulting in eight sub volumes: Low-Low-Low, Low-Low-High, Low-High-Low, Low-High-High, High-Low-Low, High-Low-High, High-High-Low and High-High-High. Similar to one dimensional case, 3D-Brownian wavelet can be constructed by 3D basic functions, $\psi_n(s, t)$. these functions are defined by

$$\psi_n(s,t) = 2\sin[(n-1/2)\pi s]\sin[(n-1/2)\pi t]. \tag{4}$$

As we know, these functions are orthonormal, so we have:

$$<\psi_n(s,t), \psi_{n-1}(s,t)>= 0,$$

$$<\psi_n(s,t), \psi_n(s,t)>= 1.$$

Some generated functions are shown in the Figures 5 and 6.

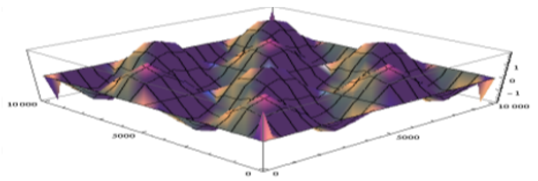

Figure 5: $\psi_3(s,t) = 2\sin[(5/2)\pi s]\sin[(5/2)\pi t]$.

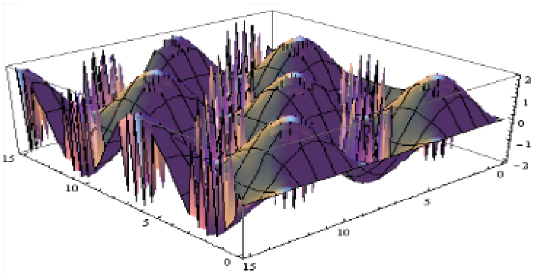

Figure 6: $\psi_4(s,t) = 2\sin[(7/2)\pi s]\sin[(7/2)\pi t]$.

The wavelet coefficients are as follow:

$$\bar{a_n} = 2\sqrt{2}a_n^2, \qquad a_n = \frac{1}{(n-1/2)\pi}. \tag{5}$$

4 Numerical Approximation of Multiple Ito-integral

There are two steps in defining a double Ito-integral. The first step is to define the integral for "off-diagonal step functions" and the second step is to approximation a function in $L^2([a,b]^2)$ by "off-diagonal step functions" then, take the limit of corresponding integrals. To motivate the notion of an "off-diagonal step function" and its necessity, consider the example of defining the double Ito-integral $\iint 1 d\beta(t)\beta d(s)$ Let $\Delta_n = \{t_0, t_1, \cdots, t_n\}$ be a partition of the interval $[0,1]$. Let, the partition of the interval square $[0,1)^2$ be as follow:

$$[0,1)^2 = \bigcup_{i,j=1}^{n} [t_{i-1}, t_i) \times [t_{j-1}, t_j).$$

Then we get the following Riemann-sum for the integral $f \equiv 1$:

$$\sum_{i,j=1}^{n} (\beta_{t_i} - \beta_{t_{i-1}})(\beta_{t_j} - \beta_{t_{j-1}}) = \sum_{i=1}^{n} (\beta_{t_i} - \beta_{t_{i-1}})^2 = \beta(1)^2.$$

4.1 Numerical approximation of Ito-integral

Now, let us consider the following integral:

$$\int_a^b f(t)d\beta(t) = \lim_{n\to\infty}\sum_{i=1}^{n} f(t_{i-1})(\beta_{t_i} - \beta_{t_{i-1}}),$$

where $f(t)$ is a Brownian motion

$$\int_a^b \beta(t)d\beta(t) = \sum_{n=1}^{\infty} \beta_{n-1}(\beta_n - \beta_{n-1}).$$

In [10] we know that $\beta(t,\omega) = \sum_{n=1}^{\infty} a_n\psi_n(t)G_n(\omega)$, where $G_n(\omega) \in [0,1]$ and $\psi_n(t), a_n$ have been defined in [14]. Without losing the generality, for $G_n(\omega)$ has the same value for both of $\beta(t,\omega)$,

$$\int_a^b \beta(t)d\beta(t) = G_n(\omega)\{\sum_{n=1}^{\infty} a_n\psi_n(t_{n-1})[\sum_{n=1}^{\infty} a_n\psi_n(t_n) - \sum_{n=1}^{\infty} a_n\psi_n(t_{n-1})]\}.$$

For a given value of $G_n(\omega)$ the following approximation can be established

$$\int_a^b \beta(t)d\beta(t) = G_n(\omega)\{-\sum_{n=1}^{\infty} a_n^2\psi_n(t_{n-1})^2\}. \tag{6}$$

Now, some examples are illustrated.

4.2 Numerical examples

By considering the above discussion two examples are given:

Example 1. We calculate the Ito-integral $\int_0^1 \beta^2(t)d\beta(t)$. We know

$$\int_a^b \beta(t)d\beta(t) = G_n(\omega)\{-\sum_{n=1}^{\infty} a_n^2\psi_n^2(t_{n-1})\}.$$

So for

$$\int_0^1 \beta^2(t)d\beta(t) = G_n(\omega)\{-\sum_{n=1}^{\infty} a_n^3\psi_n^3(t_{n-1})\},$$

for $G_n(\omega) = 0.1$,

$$0.1\{-\sum_{n=1}^{\infty} a_n^2\psi_n^2(t_{n-1})[\sum_{n=1}^{\infty} a_n\sqrt{2}\sin[(n-1/2)\pi t] - \sum_{n=1}^{\infty} a_n\sqrt{2}\sin[((n-1)1/2)\pi t]\}.$$

Now for different random value $G_n(\omega)$, numerical solutions of Ito-integral are shown in Table 1.

Table 1: Numerical solutions of Ito-integral for different random value $G_n(\omega)$.

$G_n(\omega)$	**Ito-integral approximation**
0.1	0.00253019
0.2	0.00506037
0.3	0.00759056
0.4	0.0101207
0.5	0.0126509
0.6	0.0151811
0.7	0.0177113
0.8	0.0202415
0.9	0.0227717
1	0.0253019

Now, consider the following integral:

$$\iint_a^b \beta(t,s,\omega)d\beta(t,\omega)d\beta(s,\omega),$$

where $\beta(t,s,\omega)$ is a 3D-Brownian motion. Notice to $\tilde{G}_n(\omega) \in [0,1]^2$ we have

$$\int_a^b \beta(t,\omega)[\int_a^b \beta(s,\omega)d\beta(s,\omega)]d\beta(t,\omega).$$

Now, some examples are given.

Example 2. We consider the Ito-integral $\iint_0^1 \beta^2(t,s,\omega)d\beta(t,\omega)d\beta(s,\omega)$. Similar to pervious algorithm for two steps and for different random value $\tilde{G}_n(\omega)$, we mentioned the numerical solutions in Table 2.

Table 2: Numerical solutions of multiple Ito-integral for different value $\tilde{G}_n(\omega)$.

$\tilde{G}_n$	**Multiple Ito-integral approximation**
0.01	6.40186×10^{-6}
0.02	0.0000256073
0.03	0.0000576166
0.04	0.000102429
0.05	0.000160045
0.06	0.000230466
0.07	0.00031369
0.08	0.000409718
0.09	0.00051855
1	0.000640186

5 Numerical Solution of Vasicek Equation

Firstly, consider a linear stochastic differential equation in the general form as follow:

$$\frac{dx}{dt} = f(x,t) + L(x,t)\omega(x,t),$$

where the initial conditions are $x(t_0) \sim N(m_0, P_0)$ and the $f(t)$ and $L(t)$ are matrix valued functions of time, and $w(t)$ is white noise. New research contributes to the development of the mathematical ways in solving of the bond price and interest rate under the Vasicek model. In modeling the indeterminable of interest rates, consider that there is a probability space (ω, F, P) with a standard filtration F_t. Under the risk-neutral measure P the short rate dynamics is defined as:

$$dx(t) = -x(t)dt + \sigma d\beta(t), \tag{7}$$

where σ is a positive constant. We have

$$\dot{x}(t) = -x(t) + \sigma\omega(t). \tag{8}$$

Now let we assume $x(t)$ is Brownian motion, and consider $a_n G_n = \tilde{a}_n$,

$$x(t) = \sum_{n=0}^{\infty} \tilde{a_n}\psi_n(t). \tag{9}$$

We get

$$x\dot{(t)} = \sum_{n=0}^{\infty} \tilde{a_n}\psi_n\dot{(t)}. \tag{10}$$

By multiplying the relation (8) by $\psi_n(t)$ and integrating the sides, we have:

$$\int_1^2 x\dot{(t)}\psi_n(t)dt = -\int_1^2 x(t)\psi_n(t)dt + \sigma\int_1^2 \psi_n(t)\omega(t)dt. \tag{11}$$

We know that $\omega(t)d(t) = d\beta(t)$. So, we get

$$\int_1^2 x(t)(\psi_n(t) - \psi_n\dot{(t)})dt = \sigma\int_1^2 \psi_n(t)d\beta(t). \tag{12}$$

Substituting (9) and (10) into (12) gives

$$\int_1^2 \tilde{a_n}\psi_n(t)(\psi_n(t) - \psi_n\dot{(t)})dt = \sigma\int_1^2 \psi_n(t)d\beta(t).$$

For $\sigma = 1$, we know $<\psi_n, \psi_n>= 1$. So,

$$2\tilde{a_n} = \int_1^2 \psi_n(t)d\beta(t). \tag{13}$$

Now, according to numerical approximation Ito-integral method, we can obtain the following relation.

$$2\tilde{a_n} = \sum_{n=1}^{\infty} a_n[\sum_{i=1}^{\infty} \psi_n(t_{i-1})\psi_n(t_i) - i^2].$$

Without losing the generality, let $t_i = 1, t_{i-1} = i-1 \quad (i = 1, 2, \cdots)$. Therefore, for $t_1 = 1$ and random value 0.1, we get

$$2\tilde{a_n} = \sum_{n=1}^{\infty} a_n[\psi_n(1)\psi_n(0) - 1].$$

For $n = 3$, the following system of equations is obtained:

$$2\tilde{a_1} = a_1[\psi_1(1)\psi_1(0) - 1],$$
$$2\tilde{a_2} = a_1[\psi_1(1)\psi_1(0) - 1] + a_2[\psi_2(1)\psi_2(0) - 1],$$
$$2\tilde{a_3} = a_1[\psi_1(1)\psi_1(0) - 1] + a_2[\psi_2(1)\psi_2(0) - 1] + a_3[\psi_3(1)\psi_3(0) - 1].$$

This can be summarized as the following matrix form.

$$\left[\begin{array}{c|c} \tilde{a_1} & 1/10\pi \\ \tilde{a_2} & 1/5\pi \\ \tilde{a_3} & 1/10\pi \end{array}\right].$$

Table 3: Ito-integral approximation for $n = 3$, $t_1 = 1$ and different values a_n.

a_n	$\tilde{a_1}$	$\tilde{a_2}$	$\tilde{a_3}$
0.1	$1/10\pi$	$1/5\pi$	$1/10\pi$
0.2	$1/5\pi$	$2/5\pi$	$1/5\pi$
0.3	$3/10\pi$	$3/5\pi$	$3/10\pi$
0.4	$2/5\pi$	$4/5\pi$	$2/5\pi$
0.5	$1/2\pi$	$1/\pi$	$1/2\pi$
0.6	$3/5\pi$	$6/5\pi$	$3/5\pi$
0.7	$7/10\pi$	$7/5\pi$	$7/10\pi$
0.8	$4/5\pi$	$8/5\pi$	$4/5\pi$
0.9	$9/10\pi$	$9/5\pi$	$9/10\pi$

Therefore, the corresponding results for another random value, i.e., $n = 3$ and $t_2 = 2$, are available. This can be summarized as the following matrix form that is given in Table 4.

Table 4: Ito-integral approximation for $n = 3$, $t_2 = 2$ and different values a_n.

a_n	$\tilde{a_1}$	$\tilde{a_2}$	$\tilde{a_3}$
0.1	$^1/_{2\pi}$	$^1/_{2\pi}$	$^1/_{2\pi}$
0.2	$^1/_{\pi}$	$^2/_{\pi}$	$^1/_{\pi}$
0.3	$^3/_{2\pi}$	$^3/_{\pi}$	$^3/_{2\pi}$
0.4	$^2/_{\pi}$	$^4/_{\pi}$	$^2/_{\pi}$
0.5	$^5/_{2\pi}$	$^5/_{\pi}$	$^5/_{2\pi}$
0.6	$^3/_{\pi}$	$^6/_{\pi}$	$^3/_{\pi}$
0.7	$^7/_{2\pi}$	$^7/_{\pi}$	$^7/_{2\pi}$
0.8	$^4/_{\pi}$	$^8/_{\pi}$	$^4/_{\pi}$
0.9	$^9/_{2\pi}$	$^9/_{\pi}$	$^9/_{2\pi}$

Finally, the numerical Ito-integral for $n = 3$ can be resulted in the following form:

$$a_1 = 0.1 \quad \rightarrow x(t_1) = \sum_{n=1}^{3} \tilde{a_n}\psi_n(t) = \frac{1}{10\pi}\psi_1(t) + \frac{1}{5\pi}\psi_2(t) + \frac{1}{10\pi}\psi_3(t).$$

$$a_2 = 0.2 \quad \rightarrow x(t_1) = \sum_{n=1}^{3} \tilde{a_n}\psi_n(t) = \frac{1}{5\pi}\psi_1(t) + \frac{2}{5\pi}\psi_2(t) + \frac{1}{5\pi}\psi_3(t).$$

Also, the other coefficients can be obtained by the equation (12). One of the advantages of this approach is that we can extend this relation and obtain the weights with constant values for $\tilde{a_n}$. Now, let we define the 3D-Vasicek equation by 3D-Brownian motion. This 3D-stochastic partial differential equation has the general form:

$$\frac{\partial^2 x(t,s)}{\partial t \partial s} = -x(t,s) + \sigma w(t,s), \tag{14}$$

where $x(t,s)$ is a 3D-Brownian motion and consider

$$x(t,s) = \sum_{n=0}^{\infty} \tilde{a_n}^2 \psi_n(t,s), \tag{15}$$

we get

$$\dot{x}(t,s) = \sum_{n=0}^{\infty} \tilde{a_n}^2 \dot{\psi_n}(t,s), \tag{16}$$

where $\psi_n(t,s)$ are basic functions of 3D-stochastic wavelet. Multiply the relation (14) by $\psi_n(t,s)$ and integrate the sides, the relation (17) is derived.

$$\iint_1^2 \dot{x}(t,s)\psi_n(t,s)dtds = -\iint_1^2 x(t,s)\psi_n(t,s)dtds + \sigma \iint_1^2 \psi_n(t,s)\omega(t,s)dtds. \tag{17}$$

We know that $\omega(t)\omega(s)dtds = d\beta(t,s)$. So, we know that

$$\iint_1^2 x(t,s)(\psi_n(t,s) - \dot{\psi_n}(t,s))dtds = \sigma \iint_1^2 \psi_n(t,s)d\beta(t)d\beta(s), \tag{18}$$

and Hence, similar to (13) and substituting into (15) and also (16) into (18) and for $\sigma = 1$, the relation (19) is resulted.

$$4\tilde{a_n}^2 = \iint_1^2 \psi_n(t,s)d\beta(t)d\beta(s). \tag{19}$$

Now, according to the numerical approximation multiple Ito-integral method

$$4\tilde{a_n}^2 = \sum_{n=1}^{\infty} a_n[\sum_{i=1}^{\infty} \psi_n(t_{i-1})\psi_n(t_i) - i^2] \sum_{n=1}^{\infty} a_n[\sum_{j=1}^{\infty} \psi_n(s_{j-1})\psi_n(s_j) - j^2],$$

the coefficients matrix Vasicek is obtained:

$$\mathbf{V}_{ij} = \frac{ia_j}{\pi} \qquad j,i = 1,2,\cdots.$$

Finally, the numerical multiple Ito-integration for $n = 3$ and t_1, s_1 takes the following form:

$$a_1 = 0.1 \quad \to x(t_1,s_1) = \sum_{n=1}^{3} \tilde{a_n}\psi_n(t,s) = \frac{1}{10\pi}\psi_1(t,s) + \frac{1}{5\pi}\psi_2(t,s) + \frac{1}{10\pi}\psi_3(t,s),$$

$$a_2 = 0.2 \quad \to x(t_1,s_1) = \sum_{n=1}^{3} \tilde{a_n}\psi_n(t,s) = \frac{1}{5\pi}\psi_1(t,s) + \frac{2}{5\pi}\psi_2(t,s) + \frac{1}{5\pi}\psi_3(t,s).$$

The numerical multiple Ito-integral for $n = 3$ and t_2, s_2 takes the following form

$$a_1 = 0.1 \quad \to x(t_2,s_2) = \sum_{n=1}^{3} \tilde{a_n}\psi_n(t,s) = \frac{1}{2\pi}\psi_1(t,s) + \frac{1}{\pi}\psi_2(t,s) + \frac{1}{\pi}\psi_3(t,s),$$

$$a_2 = 0.2 \quad \to x(t_2,s_2) = \sum_{n=1}^{3} \tilde{a_n}\psi_n(t,s) = \frac{1}{\pi}\psi_1(t,s) + \frac{2}{\pi}\psi_2(t,s) + \frac{1}{\pi}\psi_3(t,s).$$

6 Numerical Results

In this section, we compare the proposed numerical solution of the 3D-Vasicek equation by using Brownian wavelets with the other existing methods for this class of problems [9], [12] path-wise variable time-stepping method, Quasi-symplectic method, Midpoint rule, and Runge-Kutta for three-dimensions. The methods are applied to some models of interest, which have a relative interest rate. In each example, we ran the proposed algorithm with a given random variable, in order to compare their efficiency. We compute the error ϵ between numerical approximation and semi exact solution of the 3D-Vasicek equation. The results are shown in Table 5.

The result of the simulation for 3D-Vasicek equation by using Brownian wavelet regarding the obtained coefficients are shown in Figures 7 and 8.

Table 5: The error between numerical approximation and semi exact solution of 3D-Vasicek.

method	$\tilde{a_n} = 0.1$	$\tilde{a_n} = 0.5$	$\tilde{a_n} = 1$
Brownian wavelet	0.0275	0.003309	0.035051
Midpoint rule	0.219	0.445	0.086
Runge-Kutta	0.187	0.784	0.046
Pathwise variable time-stepping	0.3647	1.00028	0.9968
Quasi-symplectic	0.1796	0.883	0.092

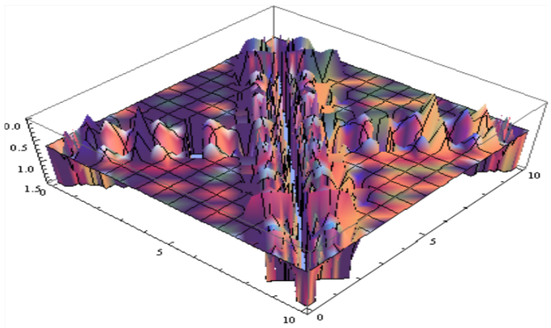

Figure 7: Simulation for 3D-Vasicek equation by using Brownian wavelet for $\tilde{a_n} = 0.2$, $n = 5$, $t_1 = 1$.

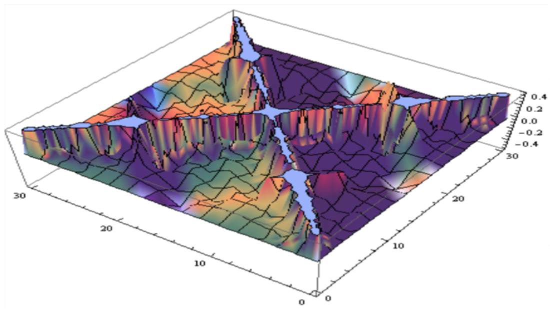

Figure 8: Simulation for 3D-Vasicek equation by using Brownian wavelet for $\tilde{a_n} = 0.5$, $n = 10$, $t_1 = 1$.

7 Conclusion

In this paper, a new approach for solving a stochastic differential equation is presented and the numerical approximate solution of the Vasicek equation is obtained. Firstly, the numerical approximation of Ito-integral based on Brownian motion is introduced. Then, the obtained algorithm evaluated the random value based on $a_n G_n = \tilde{a}_n$, and this procedure is applicable for solving multiple Ito-integral. Afterward, the 1D and 3D-Brownian wavelets based on coefficients Brownian motion are introduced. By using the obtained results, a system of linear and nonlinear equations of coefficients Brownian motion is obtained, such that by solving this system, the approximate solution of the Vasicek equation is obtained. In the end, some numerical examples are given.

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