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Research Article

A Numerical Formulation for N-Dimensional Wave Equations Using Shearlets

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Abstract. A shearlet frame approach is used to solve n-dimensional wave equations numerically. By the presented procedure, the shearlet coefficients are obtained via separate time-independent partial differential equations. The proposed method has the advantage of separation of spatial and temporal parameters. The issues of convergence and best approximation are also discussed.

Keywords. Shearlet frame, Wave equation, Shearlet coefficient

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1 Introduction and Preliminaries

The wave equation is categorized as a hyperbolic second-order partial differential equation (PDE). This kind of equations appear in different scientific fields. Several numerical techniques have so far been developed for the solution of these equations. Among these techniques, finite difference methods [16, 15], finite element methods [19], spectral methods [7, 18], etc. can be mentioned. Amongst the numerical approaches, the wavelet-based numerical methods have been developed and have widely used to solve the various types of wave equations in different dimensions, for example, see [11, 10]. Curvelets have also been employed for the solution of PDE problems, see e.g. [15, 17]. Shearlets are newer representation systems that are equipped with a rich mathematical structure similar to wavelets [13, 14]. In fact, the theory and algorithms of shearlets can be carried over the continuous wavelet transform. The continuous shearlet transform is based on special affine systems generated by one single function $\psi \in L^2(\mathbb{R}^2)$. Moreover, compared with wavelets, the continuous shearlet transform has a coherent matrix structure for n-dimensions so that it is useful for solving the higher-dimensional PDEs [1].

In this paper, by making use of shearlet properties and borrowing Kaiser's idea in [9], (see also[1]) the wave equation is transformed into the shearlet domain. For this purpose, a new shearlet ψ, for the n-dimensional case in $L^2(\mathbb{R}^n)$ is defined and it is shown that the shearlet family $\{\psi_{j,k,m}(\cdot)\}_{j,k,m}$ which is generated by ψ is a Parseval frame for $L^2(\mathbb{R}^n)$. For a better understanding of the approach and as an example, the formulation is presented in the case of two dimensions and its merits in comparison with the curvelet numerical methods are pointed. As it will be noticed, in the suggested approach the unknown function is expanded by using shearlet frames. Then by employing Fourier transform and the Plancherel theorem [6, Theorem 4.25], as well as properties of shearlets, each unknown coefficient of the expansion is obtained by solving far simpler separate time-independent PDEs. The main advantage of this approach in comparison with similar numerical methods is that there is no need to solve a set of simultaneous equations. Finally, the convergence of the presented method is discussed, in which the ideas of [3, 4] are used. Furthermore, the issues of the best approximation are also discussed.

The paper is organized as follows. In the rest of this section, some necessary definitions and theorems are explained. Section 2 is devoted to the development of n-dimensional formulation. In Section 3, an example of a two-dimensional wave problem is presented. Convergence and best approximation analysis are fully studied in Section 4. Conclusions and merits of the approach are concisely discussed in Section 5.

Firstly, required notation and definitions related to shearlets are mentioned. Let $\{\psi_{j,k,m}(\cdot)\}_{j,k,m}$ be a family of shearlets in n-dimensions as

$$\psi_{j,k,m}(\cdot) = |\det A_{2^j}|^{-\frac{1}{2}} \psi(A_{2^j}^{-1} S_k^{-1}(\cdot - m)), \tag{1}$$

where $j \in \mathbb{Z}, K = (k, \cdots, k) \in \mathbb{Z}^{n-1}, m \in \mathbb{Z}^n$ and

$$A_{2^j} = \begin{bmatrix} 2^j & 0_{n-1}^T \\ 0_{n-1} & 2^{\frac{j}{2}} I_{n-1} \end{bmatrix}, \qquad S_k = \begin{bmatrix} 1 & K^T \\ 0_{n-1} & I_{n-1} \end{bmatrix}, \tag{2}$$

and $\psi \in L^2(\mathbb{R}^n)$ is admissible in the sense that

$$C_\psi = \int_{\mathbb{R}^n} \frac{|\hat{\psi}(\xi)|^2}{|\xi_1|^2} d\xi_1 \cdots d\xi_n < \infty. \tag{3}$$

Definition 1. Let $\psi_1 \in L^2(\mathbb{R})$ be an admissible wavelet with $\hat{\psi}_1 \in C^\infty(\mathbb{R})$ and supp $\hat{\psi}_1 \subseteq [-2,-\frac{1}{2}] \cup [\frac{1}{2},2]$. Consider $\psi_2 \in L^2(\mathbb{R}^{n-1})$ as a function such that $\hat{\psi}_2 \in C^\infty(\mathbb{R}^{n-1})$ and supp $\hat{\psi}_2 \subseteq [-1,1]^{n-1}$. Then the function $\psi \in L^2(\mathbb{R}^n)$ defined by

$$\hat{\psi}(\xi) = \hat{\psi}(\xi_1,\tilde{\xi}) = \hat{\psi}_1(\xi_1) \cdot \hat{\psi}_2(\frac{\tilde{\xi}}{\xi_1}), \tag{4}$$

where $\tilde{\xi} = (\xi_2, \cdots, \xi_n)$, is an admissible shearlet. Let $\psi_1 \in L^2(\mathbb{R})$ satisfiy the discrete Calderon's condition

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}_1(2^{-j}\xi)|^2 = 1,$$

with $\hat{\psi}_1 \in C^\infty(\mathbb{R})$ and supp $\hat{\psi}_1 \subseteq [-2,-\frac{1}{2}] \cup [\frac{1}{2},2]$. Let $\psi_2 \in L^2(\mathbb{R}^n)$ be a bump function such that for all $\xi \in [-1,1]^{n-1}$,

$$\sum_{k=-1}^{1} |\hat{\psi}_2(\xi + k)|^2 = 1,$$

where $\hat{\psi}_2 \in C^\infty(\mathbb{R}^{n-1})$ and supp $\hat{\psi}_2 \subseteq [-1,1]^{n-1}$.

For $n = 2$ the shearlet ψ, defined by (4) is called a classical shearlet [13]. Making use of the above definition, we are now going to costruct a more general n-dimensional case of the classical shearlet. For this purpose, define ψ by (4), where

$$\hat{\psi}_1(\xi) = \sqrt{b^2(2\xi) + b^2(\xi)},$$

$$b(\xi) := \begin{cases} \sin(\frac{\pi}{2}v(|\xi|-1)), & 1 \le |\xi| \le 2, \\ \cos(\frac{\pi}{2}v(\frac{1}{2}|\xi|-1)), & 2 < |\xi| \le 4, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$v(x) = \begin{cases} o, & ,x < 0, \\ 2x^2, & ,0 \le x < \frac{1}{2}, \\ 1-2(1-x)^2, & ,\frac{1}{2} \le x < 1, \\ 1, & x > 1. \end{cases} \tag{5}$$

Assume $I = \{i_1, i_2, \cdots, i_{n_i}\} \subseteq \{2,3,\cdots,n\}$ and $J = \{j_1, j_2, \cdots, j_{n_j}\} \subseteq \{2,3,\cdots,n\}$ such that $I \cap J = \emptyset$, $\hat{\psi}_2(\tilde{\xi})$ is now defined as

$$\hat{\psi_2}^2(\tilde{\xi}) = \hat{\psi_2}^2(\xi_2,\xi_3,\cdots,\xi_n)$$

$$= v(1-\xi_{i_1})\cdots v(1-\xi_{i_{n_i}})v(1+\xi_{j_1})\cdots v(1+\xi_{j_{n_j}}),$$

where v is the same as (5) and it is assumed $\xi_{i_{n'}} \geq 0$ for $n' = 1, 2, \cdots, n_i$, $\xi_{j_{n''}} \leq 0$ for $n'' = 1, 2, \cdots, n_j$, such that $n_i + n_j = n-1$. By these assumptions, we will show that $\hat{\psi}_2$ satisfies the conditions in Definition 1. To do this, considering $\tilde{k} = (k_2, k_3, \cdots, k_n)$ where $k_\alpha \in \mathbb{Z}$ and $|k_\alpha| \leq 1$ for each $\alpha \in \{2, 3, \cdots, n\}$ and noting that j is the scale parameter, it follows

$$\begin{aligned}
&\sum_{k_2}\sum_{k_3}\cdots\sum_{k_n}|\hat{\psi}_2(\tilde{\xi}+\tilde{k})|^2 \\
&\quad = \sum_{k_{i_1}}\sum_{k_{i_2}}\cdots\sum_{k_{n_j}} v(1-\xi_{i_1}+k_{i_1})\cdots v(1-\xi_{i_{n_i}}+k_{i_{n_i}}) \\
&\qquad\qquad v(1+\xi_{j_1}+k_{j_1})\cdots v(1+\xi_{j_{n_j}}+k_{j_{n_j}}) \\
&\quad = \sum_{k_{i_1}} v(1-\xi_{i_1}+k_{i_1})\cdots\sum_{k_{i_{n_i}}} v(1-\xi_{i_{n_i}}+k_{i_{n_i}}) \\
&\qquad\qquad \sum_{k_{j_1}} v(1+\xi_{j_1}+k_{j_1})\cdots\sum_{k_{j_{n_j}}} v(1+\xi_{j_{n_j}}+k_{j_{n_j}}).
\end{aligned}$$

One should note that according to [8, Section 2], each of the above summations are equal to 1, therefore

$$\sum_{k_2=-1}^{1}\sum_{k_3=-1}^{1}\cdots\sum_{k_n=-1}^{1}\hat{\psi}_2(\tilde{\xi}+\tilde{k})|^2 = 1. \tag{6}$$

Moreover, by (6) and [8, Section 2] it can be easily seen that

$$\sum_{j,k}|\hat{\psi}_{j,k}(\xi)|^2 = 1, \tag{7}$$

where $\xi = (\xi_1, \cdots, \xi_n)$, $\quad k = (k_1, \cdots, k_n)$.

In the next step, we have the following proposition whose proof is similar to [13, Section 5.1, Proposition 2] and so is omitted.

Proposition 1. The shearlet system $\{\psi_{j,k,m}\}_{j,k,m}$ defined by (1) with ψ as in Definition 1, is a Parseval frame for $L^2(\mathbb{R}^n)$.

Now, borrowing the idea of Kaiser in [9, Chapter 9] for separation of spatial and temporal variables, we define $\tilde{\psi}_{j,k,m}$ as follows

$$\tilde{\psi}_{j,k,m} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$$

$$\hat{\tilde{\psi}}_{j,k,m}(\xi, t) = \hat{\psi}^{\pm}_{j,k,m}(\xi)e^{\pm i|\xi|ct}, \tag{8}$$

where ψ is a classical shearlet. Then $\{\hat{\tilde{\psi}}_{j,k,m}(\cdot)\}_{j,k,m}$ is also a Parseval frame for $L^2(\mathbb{R}^n)$. This can be proved in a very similar fashion as Proposition 1. Since $\{\tilde{\psi}_{j,k,m}\}_{j,k,m}$ is a Parseval frame for $L^2(\mathbb{R}^n)$, we can write $f \in L^2(\mathbb{R}^n)$ as ([5, chapter 5])

$$f(x,t) = \sum_{j,k,m}\langle f, \tilde{\psi}_{j,k,m}\rangle\tilde{\psi}_{j,k,m}(x,t). \tag{9}$$

We denote the shearlet coefficients $\langle f, \tilde{\psi}_{j,k,m}\rangle$ by $C_{j,k,m}$. In the next section, we present a method for solving n-dimensional wave equations with shearlet frames (8).

2 Solution Procedure by Shearlet Frames

In this section, a method for solving n-dimensional homogeneous wave equations using shearlet frames is suggested. First, we consider n-dimensional wave equations as

$$u_{tt}(x,t) = c'^2 \Delta u(x,t), \quad x = (x_1, x_2, \cdots, x_n), \ 0 \le x_i \le a_i, \ i = 1, \cdots, n, \tag{10}$$

where $\Delta u = \sum_{i=1}^{n} (\frac{\partial^2 u}{\partial x_i^2})$, $a_i \in \mathbb{R}$ and c' is the constant wave speed in $\mathbb{R}$.

Consider

$$\begin{aligned} u(x,t) &= \textstyle\sum_{j,k,m} C_{j,k,m} \tilde{\psi}_{j,k,m}(x,t), \\ \Delta u(x,t) &= \textstyle\sum_{j,k,m} C^{\Delta}_{j,k,m} \tilde{\psi}_{j,k,m}(x,t), \end{aligned} \tag{11}$$

in which

$$C_{j,k,m} = \langle u(x,t), \tilde{\psi}_{j,k,m}(x,t) \rangle, \quad C^{\Delta}_{j,k,m} = \langle \Delta u(x,t), \tilde{\psi}_{j,k,m}(x,t) \rangle. \tag{12}$$

Substituting (11) to (10) and then applying the Fourier transform and noticing the Fourier transform of the drivative, we have

$$(i|\xi_{n+1}|)^2 \sum_{j,k,m} C_{j,k,m} \hat{\tilde{\psi}}_{j,k,m}(\xi,t) = c'^2 \sum_{j,k,m} C^{\Delta}_{j,k,m} \hat{\tilde{\psi}}_{j,k,m}(\xi,t). \tag{13}$$

So

$$\sum_{j,k,m} \left[-|\xi_{n+1}|^2 C_{j,k,m} - c'^2 C^{\Delta}_{j,k,m} \right] \hat{\tilde{\psi}}_{j,k,m} = 0. \tag{14}$$

By definition of $C_{j,k,m}$ and $C^{\Delta}_{j,k,m}$, we obtain

$$\sum_{j,k,m} \left[\langle -|\xi_{n+1}|^2 u - c'^2 \Delta u, \hat{\tilde{\psi}}_{j,k,m} \rangle \right] \hat{\tilde{\psi}}_{j,k,m} = 0. \tag{15}$$

Since $\{\hat{\tilde{\psi}}_{j,k,m}\}$ is a Parseval frame for $L^2(\mathbb{R}^n)$,

$$\sum_{j,k,m} \langle -|\xi_{n+1}|^2 u - c'^2 \Delta u, \hat{\tilde{\psi}}_{j,k,m} \rangle \hat{\tilde{\psi}}_{j,k,m} = -|\xi_{n+1}|^2 u - c'^2 \Delta u. \tag{16}$$

So

$$-|\xi_{n+1}|^2 u - c'^2 \Delta u = 0.$$

Then

$$\langle -|\xi_{n+1}|^2 u - c'^2 \Delta u, \hat{\tilde{\psi}}_{j,k,m} \rangle = 0, \qquad \forall j,k,m.$$

Thus

$$-|\xi_{n+1}|^2 C_{j,k,m} - c'^2 C^{\Delta}_{j,k,m} = 0. \tag{17}$$

Using the Plancherel Theorem, we get

$$
\begin{aligned}
C_{j,k,m} &= \langle u, \tilde{\psi}_{j,k,m} \rangle \\
&= \langle \hat{U}, \hat{\hat{\psi}} \rangle \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{U}(\xi,t).\overline{\hat{\hat{\psi}}_{j,k,m}(\xi,t)} d\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{U}^{\pm}(\xi) e^{\pm i|\xi|ct}.\overline{\hat{\psi}^{\pm}_{j,k,m}(\xi) e^{\pm i|\xi|ct}} d\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{U}^{\pm}(\xi).\overline{\hat{\psi}^{\pm}_{j,k,m}(\xi)} d\xi,
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
C^{\Delta}_{j,k,m} &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{\Delta U}.\overline{\hat{\hat{\psi}}}_{j,k,m}(\xi,t) d\xi \\
&= -\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (\xi_1^2 + \cdots + \xi_n^2) \hat{U}^{\pm}(\xi).\overline{\hat{\psi}^{\pm}_{j,k,m}(\xi)} d\xi \\
&= -\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (\xi_1^2 + \cdots + \xi_n^2) \hat{U}^{\pm}(\xi).\overline{\hat{\psi}^{\pm}_{j,k}(\xi)} e^{i\langle m,\xi \rangle} d\xi.
\end{aligned}
$$

Changing the variables ξ in (18) to $S^T_{-k} A_{2^{-j}} \xi$, we obtain

$$
C_{j,k,m} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} A_{2^j} S^T_k \left(\hat{U}^{\pm}(\xi).\overline{\hat{\psi}^{\pm}}_{j,k}(\xi) \right) e^{i\langle m,\xi \rangle} d\xi,
$$

$$
C^{\Delta}_{j,k,m} = -\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |A_{2^j} S^T_k \xi|^2 A_{2^j} S^T_k \left(\hat{U}^{\pm}(\xi).\overline{\hat{\psi}^{\pm}_{j,k}}(\xi) \right) e^{i\langle m,\xi \rangle} d\xi.
$$

For simplicity, we consider $\Gamma := A_{2^j} S^T_k \left(\hat{U}^{\pm}(\xi).\overline{\hat{\psi}^{\pm}_{j,k}(\xi)} \right)$. Hence $C_{j,k,m}$ and $C^{\Delta}_{j,k,m}$ can be rewritten as

$$
C_{j,k,m} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \Gamma e^{i\langle m,\xi \rangle} d\xi, \tag{19}
$$

and

$$
C^{\Delta}_{j,k,m} = -\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |A_{2^j} S^T_k \xi|^2 \Gamma e^{i\langle m,\xi \rangle} d\xi, \tag{20}
$$

where

$$
\begin{aligned}
|A_{2^j} S^T_k \xi|^2 = [2^{2j} + k^2 2^j]\xi_1^2 + 2^j[\xi_2^2 + \cdots + \xi_n^2] \\
+ 2^{j+1} k[\xi_1\xi_2 + \xi_1\xi_3 + \cdots + \xi_1\xi_n].
\end{aligned} \tag{21}
$$

Denoting the right-hand side of (21) by $\Theta(\xi)$, $C^{\Delta}_{j,k,m}$ can be rewritten as

$$
C^{\Delta}_{j,k,m} = -\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} [\Theta(\xi)] \Gamma e^{i\langle m,\xi \rangle} d\xi.
$$

It can be seen that $C^{\Delta}_{j,k,m}$ is a combination of the following terms

$$\begin{aligned} C^{\Delta}_{k_1} &= 2^j[(2^j+k^2)\frac{\partial^2 C_{j,k,m}}{\partial m_1^2}+\frac{\partial^2 C_{j,k,m}}{\partial m_2^2}+\cdots+\frac{\partial^2 C_{j,k,m}}{\partial m_n^2}],\\ C^{\Delta}_{k_1k_n} &= 2^{j+1}k[\frac{\partial^2 C_{j,k,m}}{\partial m_1\partial m_2}+\frac{\partial^2 C_{j,k,m}}{\partial m_1\partial m_3}+\cdots+\frac{\partial^2 C_{j,k,m}}{\partial m_1\partial m_n}]. \end{aligned} \tag{22}$$

Replacing (22) in (17) leads to

$$-|\xi_{n+1}|^2 C_{j,k,m} - c'^2\left[C^{\Delta}_{k_1} + C^{\Delta}_{k_1}k_n\right] = 0. \tag{23}$$

For each j,k,m, (23) is a time-independent PDE, which can be solved by some common methods such as finite difference and pseudo-spectral methods to find the coefficients $C_{j,k,m}$. In the sequel, we present an example.

Example 1. [2] Consider the two-dimensional wave equation

$$u_{tt} = c'^2(\triangle u), \qquad 0\le x<a, \quad 0\le y<b, \tag{24}$$

where $\triangle u = u_{xx}+u_{yy}$, with initial and boundary conditions

$$\begin{cases} u|_{t=0} = f_1(x,y), & \frac{\partial u}{\partial t}|_{t=0} = f_2(x,y),\\ u(0,y,t) = u(a,y,t) = 0, & 0\le y\le b, \quad t\ge 0,\\ u(x,0,t) = u(x,b,t) = 0, & 0\le x\le a, \quad t\ge 0. \end{cases} \tag{25}$$

Applying the shearlet frame approach as was presented in the previous section to (24), the shearlet coefficients $C_{j,k,m}$ can be obtained by (23) as follows

$$-|\xi_3|^2 C_{j,k,m} - c'^2\left[C^{\Delta}_{k_1} + C^{\Delta}_{k_1k_2}\right] = 0, \tag{26}$$

where

$$\begin{aligned} C^{\Delta}_{k_1} &= 2^j[(2^j+k^2)(\frac{\partial^2 C_{j,k,m}}{\partial m_1^2})+\frac{\partial^2 C_{j,k,m}}{\partial m_2^2}],\\ C^{\Delta}_{k_1,k_2} &= 2^j(2k)(\frac{\partial^2 C_{j,k,m}}{\partial m_1\partial m_2}). \end{aligned} \tag{27}$$

Replacing (27) to (26) leads to

$$\begin{aligned} -|\xi_3|^2 C_{j,k,m} = c'^2 2^j[(2^j+k^2)(\frac{\partial^2 C_{j,k,m}}{\partial m_1^2})+(\frac{\partial^2 C_{j,k,m}}{\partial m_2^2})\\ +(2k)(\frac{\partial^2 C_{j,k,m}}{\partial m_1\partial m_2})]. \end{aligned} \tag{28}$$

Put $A := c'^2 2^j(2^j+k^2)$, $B := c'^2 k 2^j$, $D := c'^2 2^j$, $F = |\xi_3|^2$. Then we have

$$B^2 - AD = k^2 2^{2j} - 2^j(2^j+k^2)2^{2j} = -2^{3j} < 0,$$

which states that (28) is an elliptic PDE with constant coefficients with respect to m_1, m_2 and can be rewritten as follows

$$A(\frac{\partial^2 C_{j,k,m}}{\partial m_1^2}) + D(\frac{\partial^2 C_{j,k,m}}{\partial m_2^2}) + 2B(\frac{\partial^2 C_{j,k,m}}{\partial m_1 \partial m_2}) + FC_{j,k,m} = 0. \tag{29}$$

By solving the above time-independent equation for each $j, k, m,$ we get the shearlet coefficients $C_{j,k,m}$. For instance, by [8], we could consider $j = 0, 1, \cdots, j_0 - 1,$ where $j_0 = [\log_2 N], \quad k = -2^j, \cdots, 2^j, \quad m = (m_1, m_2),$ where $m_1, m_2 = 0, 1 \cdots, N-1, \quad N \in \mathbb{N}.$

It is worth noting that in comparison with the curvelet-based method in [15, 17], in which the two-dimensional wave equation is transformed into a time-dependent PDE, our time-independent method is of great advantage.

3 Convergence Analysis

In this section, convergence and best approximation analysis of the shearlet frame approach is discussed. In Section 2, it is observed that $\{\hat{\hat{\psi}}_{j,k,m}\}_{j,k,m}$ is a Parseval frame for $L^2(\mathbb{R}^n)$. By [5, Theorem 5.1.1] and noting the equivalence of items (i) and (vii) in that theorem, since

$$\sum_i \langle f, \hat{\hat{\psi}}_{j,k,m} \rangle \hat{\hat{\psi}}_{j,k,m} = 0, \tag{30}$$

implies

$$\langle f, \hat{\hat{\psi}}_{j,k,m} \rangle = 0, \qquad \forall j, k, m, \qquad (\text{by } (17)) \tag{31}$$

so $\hat{\hat{\psi}}_{j,k,m}$ is a Riesz basis. In addition, in Lemma 1, we have made use of the linear indepdency, due to having Riesz basis and noting [5, Proposition 5.1.2]. By employing the Gram-Schmidt process we are able to construct an orthonormal basis from a Riesz basis.

Let $\{\hat{\hat{\psi}}_{j',k',m'}\}_{j',k',m'}$ be an orthonormal basis for $L^2(\mathbb{R}^n)$. We define P_M as follows

$$P_M = \text{span}\{\hat{\hat{\psi}}_{j',k',m'}\}_{j',k',m'},$$

where $j' = 0, 1, \cdots, j_0 - 1, \quad j_0 = [\log_2 N], \quad k = -2^{j'}, \cdots, 2^j, \quad m_1', m_2' = 0, 1, \cdots, N$ and $M = j_0 \times 2^{j'+1} \times N^2$. Setting $l = (j, k, m)$ for simplicity, P_M can be rewritten as

$$P_M = \text{span}\{\hat{\hat{\psi}}_l\}_l. \tag{32}$$

In the rest of this section, borrowing an idea of [3], the convergence of the proposed method is proved. The following results hold for the n-dimensional case in $L^2(\mathbb{R}^n)$. For the sake of convenience, we state and prove them in dimension two. To start, it is noted that

$$\begin{aligned}\hat{U}(\xi,t) &= \sum_l C_l \hat{\hat{\psi}}_l(\xi_1,\xi_2,t) \\ &= \sum_l C_l \hat{\psi}_l^{\pm}(\xi_1,\xi_2) e^{\mp i|\xi|ct} \\ &= e^{\pm i|\xi|ct} \sum_l C_l \hat{\psi}_l^{\pm}(\xi_1,\xi_2),\end{aligned}$$

denoting $\sum_l C_l \hat{\psi}_l^{\pm}(\xi_1,\xi_2)$ by $\hat{U}_0(\xi_1,\xi_2)$, we define $\hat{U}_M$ as

$$\begin{aligned}\hat{U}_M &= e^{\mp i|\xi|ct} \sum_{l=1}^{M} C_l \hat{\hat{\psi}}_l(\xi_1,\xi_2) \\ &= e^{\mp i|\xi|ct} \hat{U}_{0M}(\xi_1,\xi_2),\end{aligned}$$

so

$$\begin{aligned}\|\hat{U}(\xi_1,\xi_2,t) - \hat{U}_M(\xi_1,\xi_2,t)\|_2 &= \|e^{\mp i|\xi|ct}\|_2 \|\hat{U}_0(\xi_1,\xi_2) - \hat{U}_{0M}(\xi_1,\xi_2)\|_2 \\ &= \|\hat{U}_0(\xi_1,\xi_2) - \hat{U}_{0M}(\xi_1,\xi_2)\|_2.\end{aligned}$$

As we observed, this norm is clearly time-independent. Hence the proof of convergence reduces to show $\|\hat{U} - \hat{U}_M\|_2 \to 0$, as $M \to \infty$. Note that for the sake of simplicity, from now on by ξ, (ξ_1,ξ_2) is meant. The following lemma which is [12, lemma 2-4-1] is useful to prove the completeness of P_M.

Lemma 1. There is a number $\eta > 0$ such that for every choice of scalars $\alpha_0, \alpha_1, \cdots, \alpha_M$, one has

$$\|\alpha_0\hat{\psi}_0 + \alpha_1\hat{\psi}_1 + \cdots + \alpha_M\hat{\psi}_M\| \geq \eta(|\alpha_0| + \cdots + |\alpha_M|).$$

Proposition 2. The space P_M in (32) is complete.

Proof. Consider the Cauchy sequence $\{\hat{W}_n(\xi)\}_{n=0}^{\infty} \in P_M$. Then each $\hat{W}_n(\xi)$ has a unique representation of the form

$$\hat{W}_n(\xi) = \sum_{i=0}^{M} \lambda_i^{(n)} \hat{\psi}_i(\xi).$$

For every $\varepsilon > 0$ there exists N' such that $\|\hat{W}_m(\xi) - \hat{W}_n(\xi)\| < \varepsilon$ when $n, m > N'$. Now by Lemma 1, there exists $\eta > 0$ such that

$$\varepsilon > \|\hat{W}_m(\xi) - \hat{W}_n(\xi)\| = \|\sum_{i=0}^{M}(\lambda_i^{(m)} - \lambda_i^{(n)})\hat{\psi}_i(\xi)\| \geq \eta \sum_{i=0}^{M} |\lambda_i^{(m)} - \lambda_i^{(n)}|.$$

Division by η gives

$$|\lambda_i^{(m)} - \lambda_i^{(n)}| \leq \sum_{i=0}^{M} |\lambda_i^{(m)} - \lambda_i^{(n)}| < \frac{\varepsilon}{\eta},$$

i.e., every sequence $\{\lambda_i^{(n)}\}_{n=0}^{\infty}$, $i = 0, 1, \cdots, M$ is Cauchy in $\mathbb{R}$. Hence it in convergent. Let λ_i denote the limit. Using this $M+1$ limits $\lambda_0, \lambda_1, \cdots, \lambda_M$, we define

$$\overline{\hat{W}}(\xi) = \sum_{i=0}^{M} \lambda_i \hat{\psi}_i(\xi).$$

Obviously $\overline{\hat{W}}(\xi) \in P_M$, so

$$\|\hat{W}_n(\xi) - \overline{\hat{W}}(\xi)\| = \|\sum_{i=0}^{M} (\lambda_i^{(n)} - \lambda_i)\hat{\psi}_i(\xi)\| \leq \sum_{i=0}^{M} |\lambda_i^{(n)} - \lambda_i| \|\hat{\psi}_i(\xi)\|.$$

Since $\lambda_i^{(n)} \to \lambda_i$, we have $\|\hat{W}_n(\xi) - \overline{\hat{W}}(\xi)\| \to 0$, i.e. $\hat{W}_n(\xi) \to \overline{\hat{W}}(\xi)$. This shows that $\{\hat{W}_n(\xi)\}_{n=0}^{\infty}$ is convergent in P_M and the proof is complete. □

Proposition 3. For every given continuous function $\hat{U}(\xi) \in L^2(\mathbb{R}^2)$ there exists a unique $\hat{U}_M(\xi) \in P_M$ such that

$$\delta = \inf_{\hat{\hat{U}} \in P_M} \|\hat{U}(\xi) - \hat{\hat{U}}(\xi)\| = \|\hat{U}(\xi) - \hat{U}_M(\xi)\|.$$

Proof. Existence: By definition of infimum, there is a sequence $\{\hat{W}_n(\xi)\}_{n=0}^{\infty}$ in P_M such that $\delta_n \to \delta$, in which $\delta_n = \|\hat{U}(\xi) - \hat{W}_n(\xi)\|$, i.e.

$$\forall \varepsilon > 0, \exists N_0 > 0 \ s.t. \ \text{ for } \ n > N_0, \ \ |\delta_n - \delta| < \varepsilon.$$

We show that $\{\hat{W}_n(\xi)\}_{n=0}^{\infty}$ is Cauchy. Writing $\hat{V}_n(\xi) = \hat{U}(\xi) - \hat{W}_n(\xi)$, we have $\|\hat{V}_n(\xi)\| = \delta_n$ and

$$\|\hat{V}_m(\xi) + \hat{V}_n(\xi)\| = 2\|\frac{1}{2}(\hat{W}_m(\xi) + \hat{W}_n(\xi)) - \hat{U}(\xi)\| \geq 2\delta.$$

Since P_M is linear, $\frac{1}{2}(\hat{W}_m(\xi) + \hat{W}_n(\xi)) \in P_M$. Also we have

$$\hat{V}_m(\xi) - \hat{V}_n(\xi) = \hat{W}_m(\xi) - \hat{W}_n(\xi).$$

Now, by the parallelogram equality,

$$\begin{aligned}\|\hat{W}_n(\xi) - \hat{W}_m(\xi)\|^2 &= \|\hat{V}_m(\xi) - \hat{V}_n(\xi)\|^2 \\ &= -\|\hat{V}_m(\xi) - \hat{V}_n(\xi)\|^2 + 2(\|\hat{V}_m(\xi)\|^2 + \|\hat{V}_n(\xi)\|^2) \\ &\leq -(2\delta)^2 + 2(\delta_m^2 + \delta_n^2) \\ &< \varepsilon^2.\end{aligned}$$

This implies that $\{\hat{W}_n(\xi)\}_{n=0}^{\infty}$ is Cauchy. Since P_M is complete, $\{\hat{W}_n(\xi)\}_{n=0}^{\infty}$ converges. Let $\hat{W}_n(\xi) \to \overline{\hat{W}}(\xi) \in P_M$. Hence we have $\|\hat{U}(\xi) - \overline{\hat{W}}(\xi)\| \geq \delta$. Also,

$$\begin{aligned}\|\hat{U}(\xi) - \overline{\hat{W}}(\xi)\| &\leq \|\hat{U}(\xi) - \hat{W}_n(\xi)\| + \|\hat{W}_n(\xi) - \overline{\hat{W}}(\xi)\| \\ &= \delta_n + \|\hat{W}_n(\xi) - \overline{\hat{W}}(\xi)\| \to \delta.\end{aligned}$$

This shows that $\|\hat{U}(\xi)-\hat{\overline{W}}(\xi)\|=\delta$.

Uniqueness: Let $\hat{\overline{W}}(\xi),\hat{W}_0(\xi)\in P_M$ satisfy

$$\|\hat{U}(\xi)-\hat{\overline{W}}(\xi)\|=\delta,$$
$$\|\hat{U}(\xi)-\hat{W}_0(\xi)\|=\delta.$$

By the parallelogram equality,

$$\begin{aligned}\|\hat{\overline{W}}(\xi)-\hat{W}_0(\xi)\|^2&=\|(\hat{\overline{W}}(\xi)-\hat{U}(\xi))-(\hat{W}_0(\xi)-\hat{U}(\xi))\|^2\\&=2\|\hat{\overline{W}}(\xi)-\hat{U}(\xi)\|^2+2\|\hat{W}_0(\xi)-\hat{U}(\xi)\|^2\\&\quad-\|(\hat{\overline{W}}(\xi)-\hat{U}(\xi))+(\hat{W}_0(\xi)-\hat{U}(\xi))\|^2\\&=4\delta^2-4\|\frac{1}{2}(\hat{\overline{W}}(\xi)+\hat{W}_0(\xi))-\hat{U}(\xi)\|^2\leq 0.\end{aligned}$$

So $\hat{\overline{W}}(\xi)=\hat{W}_0(\xi)$.

Orthogonality: Let $0\neq\hat{W}_1(\xi)\in P_M$ such that

$$\langle\hat{Z}(\xi),\hat{W}_1(\xi)\rangle=\gamma\neq 0,$$

where $\hat{Z}(\xi)=\hat{U}(\xi)-\hat{\overline{W}}(\xi)$. For any scalar η,

$$\begin{aligned}\|\hat{Z}(\xi)-\eta\hat{W}_1(\xi)\|^2&=\langle\hat{Z}(\xi)-\eta\hat{W}_1(\xi),\hat{Z}(\xi)-\eta\hat{W}_1(\xi)\rangle\\&=\|\hat{Z}(\xi)\|^2-\overline{\eta}\gamma-\eta(\overline{\gamma}-\overline{\eta}\|\hat{W}_1(\xi)\|^2),\end{aligned}$$

choosing $\overline{\eta}=\frac{\overline{\gamma}}{\|\hat{W}_1(\xi)\|}$ yields

$$\begin{aligned}\|\hat{Z}(\xi)-\eta\hat{W}_1(\xi)\|^2&=\|\hat{Z}(\xi)\|^2-\frac{\overline{\gamma}}{\|\hat{W}_1(\xi)\|}\\&=\gamma^2-\frac{\overline{\gamma}}{\|\hat{W}_1(\xi)\|}\geq\gamma^2,\end{aligned}$$

(i.e., $\|\hat{Z}(\xi)-\eta\hat{W}_1(\xi)\|^2\geq\gamma$). But this is impossible by the definition of γ. So

$$\langle\hat{Z}(\xi),\hat{\hat{U}}(\xi)\rangle=0,\quad\forall\hat{\hat{U}}\in P_M.$$

Now, it is shown that $\hat{\overline{W}}(\xi)=\hat{U}_M(\xi)$. From the proof provided above that indicates $\hat{\overline{W}}(\xi)$ is the best approximation for $\hat{U}(\xi)$, it follows, $\forall i=0,1,\cdots,M$,

$$\langle\hat{U}(\xi)-\hat{\overline{W}}(\xi),\hat{\psi}_i(\xi)\rangle=0,$$

$$\begin{aligned}\langle\hat{\overline{W}}(\xi)-\hat{U}_M(\xi),\hat{\psi}_i(\xi)\rangle&=\langle\hat{U}(\xi)-(\hat{U}(\xi)-\hat{\overline{W}}(\xi))-\hat{U}_M(\xi),\hat{\psi}_i(\xi)\rangle\\&=\langle\hat{U}(\xi),\hat{\psi}_i(\xi)\rangle-\langle(\hat{U}(\xi)-\hat{\overline{W}}(\xi)),\hat{\psi}_i(\xi)\rangle\\&\quad-\langle\hat{U}_M(\xi),\hat{\psi}_i(\xi)\rangle\\&=c_i-0-c_i\\&=0.\end{aligned}$$

Therefore $\hat{\overline{W}}(\xi)-\hat{U}_M(\xi)=0$. This shows that $\hat{U}_M(\xi)=\hat{\overline{W}}(\xi)$ and the proof is complete. □

The forthcoming theorems state the decay of the shearlet coefficients and convergence of the best approximation.

Theorem 1. The shearlet coefficients c_i in the approximate solution $\hat{U}_M(\xi) = \sum_{i=1}^{M} c_i \hat{\psi}_i(\xi)$ decay, when M increases.

Proof. With the properties of the orthogonality of $\hat{\psi}_i(\xi)$, we obtain

$$\begin{aligned}\|\hat{U}(\xi) - \hat{U}_M(\xi)\|_2^2 &= \int_{\mathbb{R}^2} |\hat{U}(\xi) - \hat{U}_M(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^2} |\hat{U}(\xi)|^2 d\xi - \int_{\mathbb{R}^2} \hat{U}(\xi)\overline{\hat{U}_M(\xi)} d\xi \\ &\quad - \int_{\mathbb{R}^2} \overline{\hat{U}(\xi)}\hat{U}_M(\xi) d\xi + \int_{\mathbb{R}^2} |\hat{U}_M(\xi)|^2 d\xi.\end{aligned}$$

Since

$$\int_{\mathbb{R}^2} \hat{U}(\xi)\overline{\hat{U}_M(\xi)} d\xi = \sum_{i=0}^{M} \overline{c_i} \int_{\mathbb{R}^2} \hat{U}(\xi)\overline{\hat{\psi}_i(\xi)} d\xi = \sum_{i=0}^{M} |c_i|^2,$$

and $\int_{\mathbb{R}^2} \overline{\hat{U}(\xi)}\hat{U}_M(\xi) d\xi = \sum_{i=0}^{M} |c_i|^2,$

$$\begin{aligned}\|\hat{U}(\xi) - \hat{U}_M(\xi)\|_2^2 &= \int_{\mathbb{R}^2} |\hat{U}(\xi)|^2 d\xi - 2\sum_{i=0}^{M} |c_i|^2 + \sum_{i=0}^{M} |c_i|^2 \\ &= \int_{\mathbb{R}^2} |\hat{U}(\xi)|^2 d\xi - \sum_{i=0}^{M} |c_i|^2.\end{aligned}$$

Hence

$$\sum_{i=0}^{M} |c_i|^2 \le \int_{\mathbb{R}^2} |\hat{U}(\xi)|^2 d\xi, \qquad \forall M \in \mathbb{N}.$$

Consequently $\sum_{i=0}^{\infty} |c_i|^2$ is convergent and $\lim_{i\to\infty} c_i = 0$. □

Theorem 2. The approximate solution $\hat{U}_M(\xi) = \sum_{i=1}^{M} c_i \hat{\psi}_i(\xi)$ converges to $\hat{U}(\xi) = \sum_{i} c_i \hat{\psi}_i(\xi)$.

Proof. Let $\hat{U}_M(\xi)$ and $\hat{U}_N(\xi)$ be approximate solutions with $N > M$. Then $\{\hat{U}_N(\xi)\}_{N=0}^{\infty}$ is a Cauchy sequence in P_M. Indeed,

$$\begin{aligned}\|\hat{U}_N(\xi)-\hat{U}_M(\xi)\|_2^2 &= \|\sum_{i=M+1}^{N} c_i\hat{\psi}_i(\xi)\|_2^2\\ &= \langle \sum_{i=M+1}^{N} c_i\hat{\psi}_i(\xi), \sum_{j=M+1}^{N} c_i\hat{\psi}_j(\xi)\rangle\\ &= \sum_{i=M+1}^{N}\sum_{j=M+1}^{N} c_i\bar{c}_i\langle\hat{\psi}_i(\xi),\hat{\psi}_j(\xi)\rangle = \sum_{i=M+1}^{N}|c_i|^2.\end{aligned}$$

Since $\sum_{i=0}^{\infty}|c_i|^2$ is convergent, $\{\hat{U}_N(\xi)\}_{N=0}^{\infty}$ is Cauchy. From completeness of P_M, one has $\hat{U}_N(\xi)\to\hat{S}(\xi)\in P_M$. It could be easily shown that

$$\langle\hat{S}(\xi)-\hat{U}(\xi),\hat{\psi}_i(\xi)\rangle = 0.$$

So $\hat{S}(\xi)-\hat{U}(\xi)=0$, i.e., $\lim_{N\to\infty}\hat{U}_N(\xi)=\hat{U}(\xi)$, where the limit is in $L^2(\mathbb{R}^2)$. By the Fourier uniqueness theorem [6, Theorem 4.33], since the Fourier transform is one-to-one, we have $\lim_{N\to\infty}U_N(\xi)=U(\xi)$. □

4 Conclusion

A shearlet frame approach for the solution of n-dimensional transient wave equations was proposed. To better clarify the method, an example in two dimensions was presented. This approach in general can be applied to other PDEs such as Poisson and heat equations. As it was shown, the unknown function was expanded via shearlet frames and its coefficients were obtained by employing Fourier transform and the Plancherel Theorem. The main merit of this approach is that for finding the unknown coefficients, there is no need to solve a system of simultaneous equations and the coefficients can be obtained from a separate time-independent PDE. This property is important since the user of the method can increase the accuracy of the solution to any desired degree by evaluating appropriate coefficients. In the last section, the issues of convergence and best approximation were also discussed.

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