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**Research Article**

# Topological Subdifferential and its Role in Nonsmooth Optimization with Quasiconvex Data

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**Abstract.** In this paper, we study nonsmooth optimization problems with quasiconvex functions using topological subdifferential. We present some necessary and sufficient optimality conditions and characterize topological pseudoconvex functions. Finally, the Mond-Weir type weak and strong duality results are stated for the problems.

**Keywords.** Quasiconvex optimization, Duality theorems, Optimality conditions, Pseudoconvex, Subdifferential.

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## 1 Introduction

A mathematical optimization problem with a quasiconvex objective function and finitely many quasiconvex constraints is called a *quasiconvex programming problem*. Problems of this type have been utilized for the modeling of topological optimization problems and some theoretical topics. (see, e,g., [3, 5, 6, 7] and therein references).
This paper deals with a nonsmooth quasiconvex programming problem. We do not assume that the data of the problem are differentiable. Thus, we replace the derivative appearing in the classical results with topological subdifferential. We refer to [10] and [11, Section 5] to see the importance of this subdifferential and its relation to other subdifferentials.
It is worth to mention that some necessary conditions of Karush–Kuhn–Tucker (KKT) type for optimality of nonsmooth quasiconvex programming problems have been presented in [7] under various subdifferentials; for instance, Greenberg–Pierskalla subdifferential, Penot subdifferential, Plastria subdifferential, and Gutiérrez subdifferential. Since there are no articles that study necessary and sufficient optimality conditions under topological subdifferential, the aim of this paper is to fill this gap as the first task.

The structure of subsequent sections of this paper is as follows: In Section 2, we establish required definitions and preliminary results which are required thereafter. In Section 3, we present the KKT type necessary and sufficient optimality conditions for nonsmooth quasiconvex problems, and as applications of proved optimality conditions, we study the duality results of the problem.

## 2 Notations and Preliminaries

In this section, we present some definitions and auxiliary results that will be needed in what follows.
For a non-empty subset $M$ of $\mathbb{R}^n$, its polar cone is defined as

$$M^0 := \{x \in \mathbb{R}^n \mid \langle x, d\rangle \leq 0, \quad \forall d \in M\},$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner-product in $\mathbb{R}^n$. Also, $\overline{M}$, $conv(M)$, and $cone(M)$ denote the closure of $M$, the convex hull of $M$, and the convex cone hull of $M$, respectively. Moreover, the attainable cone and the interior cone of $M$ at $x_0 \in M$ are respectively defined as [2]:

$$\mathcal{A}(M,x_0) := \Big\{z \in \mathbb{R}^n \mid \forall t_k \downarrow 0, \exists z_k \to z, x_0 + t_k z_k \in M \quad \forall t \in \mathbb{N}\Big\},$$

$$\mathcal{I}(M,x_0) := \Big\{z \in \mathbb{R}^n \mid \exists K > 0,\ \forall t_k \downarrow 0, \forall z_k \to z, x_0 + t_k z_k \in M,\ \forall k \geq K\Big\}.$$

Observe that $\mathcal{A}(M,x_0)$ is always a non-empty closed cone ([2, Theorem 3.4.3]) and $\mathcal{I}(M,x_0)$ is always a open cone ([2, Theorem 3.4.6]).

**Theorem 1.** [2, Theorem 3.4.10] Let $C \subseteq \mathbb{R}^n$ be a convex set. If $\mathcal{I}(M,x_0) \neq \emptyset$, then

$$\overline{\mathcal{I}(M,x_0)} = \mathcal{A}(M,x_0).$$

Let $C$ be a convex subset of $\mathbb{R}^n$ and $x_0 \in C$. The normal cone of $C$ at $x_0$ is denoted by $N(C,x_0)$, i.e.,

$$N(C,x_0) := \Big\{y \in \mathbb{R}^n \mid \langle y, x - x_0\rangle \leq 0 \qquad \forall x \in C\Big\}.$$

The zero vector of $\mathbb{R}^n$ is denoted by $0_n$. If $K \subseteq \mathbb{R}^n$ is a convex cone, [2, Theorem 2.3.3] and the fact that $N(K^0, 0_n) = (K^0)^0$ imply that

$$N(K^0, 0_n) = \overline{K}. \tag{1}$$

We recall from [11] that the topological (or incident, or upper epi-) directional derivative of a given function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at $\hat{x} \in dom\varphi := \{x \in \mathbb{R}^n \mid \varphi(x) \in \mathbb{R}\}$ is defined by

$$\varphi^{\flat}(\hat{x}; v) := \sup_{\varepsilon>0} \limsup_{t\downarrow 0} \inf_{\|u-v\|<\varepsilon} \frac{\varphi(\hat{x}+tu)-\varphi(\hat{x})}{t}, \qquad \forall v \in \mathbb{R}^n.$$

Also, the epigraph of $\varphi$ is denoted by $E_\varphi$,

$$E_\varphi := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid \varphi(x) \le r\}.$$

The following theorem presents an important property of topological directional derivative.

**Theorem 2.** [11, Page 263] & [2, Page 245] The epigraph of function $v \to \varphi^{\flat}(\hat{x}; v)$ coincides to attainable cone of $E_\varphi$ at $(\hat{x}, \varphi(\hat{x}))$. In other word,

$$E_{\varphi^{\flat}(\hat{x};.)} = \mathcal{A}(E_\varphi, \hat{x}_\varphi),$$

in which, $\hat{x}_\varphi := (\hat{x}, \varphi(\hat{x}))$.

We say that $\varphi$ is regular at $\hat{x}$ if the function $v \to \varphi^{\flat}(\hat{x}; v)$ is convex (this type of functions, without special naming, have been studied in [9, Proposition 6.5]). According to Theorem 2 and [2, Theorem 2.5.1], we conclude that $\varphi$ is regular at $\hat{x}$ iff $\mathcal{A}(E_\varphi, \hat{x}_\varphi)$ is a convex set. Also, if $\varphi$ is convex and $\hat{x} \in dom\varphi$, it is regular at $\hat{x}$ by [12, Theorem 23.1] (see also [2, Theorem 2.5.1]). Let $r \in \mathbb{R} \cup \{+\infty\}$ be given. The $r$-sublevel set of $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is denoted by $[\varphi \le r]$,

$$[\varphi \le r] := \{x \in \mathbb{R}^n \mid \varphi(x) \le r\}.$$

A function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be quasiconvex, if for each $r \in \mathbb{R} \cup \{+\infty\}$ its corresponding $r$-sublevel set $[\varphi \le r]$ is convex. We can see that $\varphi$ is quasiconvex if and only if ([2, Theorem 2.10.1])

$$\varphi\Big(\mu x + (1-\mu)y\Big) \le \max\Big\{\varphi(x), \varphi(y)\Big\}, \qquad \forall x, y \in \mathbb{R}^n, \quad \forall \mu \in [0,1]. \tag{2}$$

Equivalently, $\varphi$ is quasiconvex if and only if

$$\Big(\varphi(y) \le \varphi(x), \quad 0 \le \mu \le 1\Big) \implies \varphi\Big(\mu x + (1-\mu)y\Big) \le \varphi(x).$$

The topological (or incident [11]) subdifferential of a $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at $\hat{x} \in dom\varphi$ is defined by

$$\partial^{\flat}\varphi(\hat{x}) := \Big\{\xi \in \mathbb{R}^n \mid \langle \xi, v \rangle \le \varphi^{\flat}(\hat{x}; v), \quad \forall v \in \mathbb{R}^n\Big\}.$$

In [10, Propositions 1, 3, 16] some extraordinary properties for $\varphi^{\flat}(\hat{x}; v)$ and $\partial^{\flat}\varphi(\hat{x})$ which shows their important rule and uniqueness, were proved. For instance, see the following results:

**Theorem 3.** [10, Proposition 1] $\varphi^{\flat}(\hat{x};.)$ is a lower semi-continuous (l.s.c. in brief) quasiconvex function when $\varphi$ is a quasiconvex function.

We are able to see ([10, Proposition 47, and Corollary 50]) that for a convex function $\phi : \mathbb{R}^n \to \mathbb{R}$, $\partial^{\flat}\phi(\hat{x})$ coincides with the Fenchel subdifferential $\partial\phi(\hat{x})$, is defined as follows:

$$\partial\phi(\hat{x}) := \Big\{\xi \in \mathbb{R}^n \mid \phi(x) \ge \phi(\hat{x}) + \Big\langle \xi, x - \hat{x} \Big\rangle, \qquad \forall\, x \in \mathbb{R}^n\Big\}.$$

It is worth to observe that $\partial\phi(\hat{x})$ is a non-empty closed convex set in $\mathbb{R}^n$ ([2, Page 373]), and if $\hat{x}$ is a minimizer of convex function $\phi$ on a convex set $C$, then ([2, Theorem 4.3.2])

$$0_n \in \partial\phi(\hat{x}) + N(C, \hat{x}). \tag{3}$$

## 3 Main Results

Throughout this paper, we shall consider the following optimization problem:

$$\text{(P):} \qquad \inf \vartheta(x), \quad \text{s.t.} \quad x \in \Omega := \Big\{x \in \mathbb{R}^n \mid \psi_j(x) \le 0 \qquad \forall j \in J := \{1, \dots, m\}\Big\},$$

where $\vartheta$ and $\psi_j$, $j \in J$ are quasiconvex functions from $\mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$.

**Remark 1.** The quasiconvexity of $\psi_j$ functions (for $j \in J$) and the fact that

$$\Omega = \bigcap_{j=1}^{m} \Big\{x \in \mathbb{R}^n \mid \psi_j(x) \le 0\Big\} = \bigcap_{j=1}^{m} [\psi_j \le 0],$$

imply that the feasible set $\Omega$ is convex. Thus, $\mathcal{A}(\Omega, \hat{x}) = \overline{\mathcal{I}(\Omega, \hat{x})}$ whenever $\mathcal{I}(\Omega, \hat{x}) \neq \emptyset$ by Theorem 1.

**Lemma 1.** Suppose that $\hat{x} \in \Omega$ is an optimal solution of (P). If $\mathcal{I}(\Omega, \hat{x}) \neq \emptyset$ and $\vartheta$ is regular at $\hat{x}$, then

$$\vartheta^{\flat}(\hat{x}; v) \ge 0, \qquad \forall v \in \mathcal{A}(\Omega, \hat{x}).$$

*Proof.* Let $d \in \mathcal{I}(\Omega, \hat{x})$. By definition of interior cone, for each sequence $d_k \to d$ and $t_k \to 0^+$, $\hat{x} + t_k d_k \in \Omega$ for all $k \in \mathbb{N}$. Thus, the optimality of $\hat{x}$ implies that $\vartheta(\hat{x} + t_k d_k) - \vartheta(\hat{x}) \ge 0$, and hence

$$\vartheta^{\flat}(\hat{x}; d) \ge 0 \qquad \forall d \in \mathcal{I}(\Omega, \hat{x}). \tag{4}$$

Considering Remark 1, for each $v \in \mathcal{A}(\Omega, \hat{x}) = \overline{\mathcal{I}(\Omega, \hat{x})}$, we can find a sequence $\{d_k\}_{k=1}^{\infty}$ in $\mathcal{I}(\Omega, \hat{x})$ converging to $v$. Taking into consideration the continuity of $\vartheta^{\flat}(\hat{x};.)$ (by the regularity of $\vartheta$) and the validity of (4), we obtain that

$$\vartheta^{\flat}(\hat{x}; v) = \lim_{k \to \infty} \vartheta^{\flat}(\hat{x}; d_k) \ge 0,$$

and the proof is complete. □

For a given $x_0 \in \Omega$, with the convention $\bigcup_{j \in \emptyset} X_j = \emptyset$, set

$$J(x_0) := \Big\{j \in J \mid \psi_j(x_0) = 0\Big\},$$
$$\mathcal{Z}(x_0) := \bigcup_{j \in J(x_0)} \partial^{\flat} \psi_j(x_0).$$

The following constraint qualification will be needed for stating the necessary optimality condition in Karush-Kuhn-Tucker (KKT) type.

**Definition 1.** Let $x_0 \in \Omega$. We say that the generalized Kuhn-Tucker constraint qualification (GKTCQ in brief) holds at $x_0$ if

$$\Big(\mathcal{Z}(x_0)\Big)^0 \subseteq \mathcal{A}(\Omega, x_0).$$

**Theorem 4.** Suppose that $\hat{x} \in \Omega$ is an optimal solution of (P), $\mathcal{I}(\Omega, \hat{x}) \neq \emptyset$, and that $\vartheta$ is regular at $\hat{x}$. If the (GKTCQ) is satisfied at $\hat{x}$, then

$$0_n \in \partial^{\flat} \vartheta(\hat{x}) + \overline{conv\Big(\mathcal{Z}(\hat{x})\Big)}.$$

*Proof.* At the first, we observe from [11, Page 263] that $\vartheta^{b}(\hat{x};0_n)$ is either 0 or $-\infty$. On the other hand, because of $0 \in \mathcal{A}(\Omega,\hat{x})$, we have $\vartheta^{b}(\hat{x};0_n) \geq 0$ by Lemma 1. Thus,

$$\vartheta^{b}(\hat{x};0_n) = 0. \tag{5}$$

Let $v \in \left(\overline{conv\left(\mathcal{Z}(\hat{x})\right)}\right)^0$. Because of $\left(\overline{conv\left(\mathcal{Z}(\hat{x})\right)}\right)^0 = \left(\mathcal{Z}(\hat{x})\right)^0$ and the (GKTCQ), we conclude $\vartheta^{b}(\hat{x};v) \geq 0$ in view of Lemma 1. By (5) we thus obtain that the following optimization problem has a local solution at $\widetilde{v} := 0_n$:

$$\begin{array}{ll} & \min\ \vartheta^{b}(\hat{x};.)(v) \\ \text{s.t.} & v \in \left(\overline{conv\left(\mathcal{Z}(\hat{x})\right)}\right)^0. \end{array}$$

Since the objective function and the constraint set of above problem are convex, by (3) we give

$$0_n \in \partial\left(\vartheta^{b}(\hat{x};.)\right)(0_n) + N\left(\left(\overline{conv\left(\mathcal{Z}(\hat{x})\right)}\right)^0, 0_n\right). \tag{6}$$

Now, we observe that $N\left(\left(\overline{conv\left(\mathcal{Z}(\hat{x})\right)}\right)^0, 0_n\right) = \overline{conv\left(\mathcal{Z}(\hat{x})\right)}$ by (1), and we have also

$$\begin{aligned} \partial\left(\vartheta^{b}(\hat{x};.)\right)(0_n) &= \left\{\xi \in \mathbb{R}^n \mid \vartheta^{b}(\hat{x};w) - \vartheta^{b}(\hat{x};0_n) \geq \left\langle \xi, w - 0_n\right\rangle \quad \forall\, w \in \mathbb{R}^n\right\} \\ &= \left\{\xi \in \mathbb{R}^n \mid \vartheta^{b}(\hat{x};w) \geq \left\langle \xi, w\right\rangle \quad \forall\, w \in \mathbb{R}^n\right\} \\ &= \partial^{b}\vartheta(\hat{x}). \end{aligned} \tag{7}$$

Hence, (6) implies that

$$0_n \in \partial^{b}\vartheta(\hat{x}) + \overline{conv\left(\mathcal{Z}(\hat{x})\right)},$$

and the proof is complete. □

We should mention that the following KKT type necessary condition is similar that one proved in [1] and [4] using Clarke subdifferential.

**Theorem 5.** (KKT necessary condition) Let $\hat{x} \in \Omega$ be an optimal solution of (P). Assume that $\vartheta$ is regular at $\hat{x}$, $\mathcal{I}(\Omega,\hat{x}) \neq \emptyset$, and the (GKTCQ) is satisfied at $\hat{x}$. Moreover, if $conv\left(\mathcal{Z}(\hat{x})\right)$ is a closed set, then there exist $\lambda_j \geq 0,\ j \in J(\hat{x})$, such that

$$0_n \in \partial^{b}\vartheta(\hat{x}) + \sum_{j\in J(\hat{x})} \lambda_j \partial^{b}\psi_j(\hat{x}), \qquad \textit{and} \qquad \sum_{j\in J(\hat{x})} \lambda_j = 1.$$

*Proof.* It follows from Theorem 4 and the closedness of $conv\left(\mathcal{Z}(\hat{x})\right)$ that

$$0_n \in \partial^{b}\vartheta(\hat{x}) + \overline{conv\left(\mathcal{Z}(\hat{x})\right)} = \partial^{b}\vartheta(\hat{x}) + conv\left(\mathcal{Z}(\hat{x})\right).$$

Thus, there exist $\xi^{\vartheta} \in \partial^{b}\vartheta(\hat{x})$ and $w \in conv\left(\mathcal{Z}(\hat{x})\right)$ such that

$$0_n = \xi^{\vartheta} + w. \tag{8}$$

On the other hand, the regularity assumption of $\psi_j$ as $j \in J(\hat{x})$ and (7) imply that $\partial^{b}\psi_j(\hat{x})$ is a closed convex set in $\mathbb{R}^n$ for $j \in J(\hat{x})$. Thus, the well-known equality (see, e.g., [12, Theorem 3.3])

$$conv\Big(\mathcal{Z}(\hat{x})\Big) = \Big\{ \sum_{j\in J(\hat{x})} \lambda_j\zeta_j \mid \sum_{j\in J(\hat{x})} \lambda_j = 1,\ \lambda_j \geq 0,\ \zeta_j \in \partial^\flat \psi_j(\hat{x}),\ j \in J(\hat{x}) \Big\},$$

implies that for each $j \in J(\hat{x})$ there exist $\zeta_j \in \partial^\flat \psi_j(\hat{x})$ and $\lambda_j \geq 0$ such that

$$w = \sum_{j\in J(\hat{x})} \lambda_j\zeta_j, \qquad \text{and} \qquad \sum_{j\in J(\hat{x})} \lambda_j = 1.$$

The last relations and (8) allow us to conclude that

$$0_n = \xi^\vartheta + \sum_{j\in J(\hat{x})} \lambda_j\zeta_j \ \in\ \partial^\flat \vartheta(\hat{x}) + \sum_{j\in J(\hat{x})} \lambda_j \partial^\flat \psi_j(\hat{x}),$$

as required. □

The following useful lemma characterizes quasiconvex functions using topological subdifferential.

**Lemma 2.** If $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a quasiconvex function, then

$$\varphi(x) \leq \varphi(x_0) \Longrightarrow \langle \xi, x - x_0 \rangle \leq 0 \qquad \forall \xi \in \partial^\flat \varphi(x_0).$$

*Proof.* Assume that $\varphi(x) \leq \varphi(x_0)$. By (2), for each $t \in (0,1)$, we have

$$\varphi\Big(x_0 + t(x - x_0)\Big) = \varphi\Big(tx + (1-t)x_0\Big) \leq \max\Big\{\varphi(x), \varphi(x_0)\Big\} = \varphi(x_0).$$

This implies $\varphi^\flat(x_0; x - x_0) \leq 0$. Thus, by the definition of incident subdifferential, we get

$$0 \geq \varphi^\flat(x_0; x - x_0) \geq \sup\Big\{\langle \xi, x - x_0\rangle \mid \xi \in \partial^\flat \varphi(x_0)\Big\}.$$

Therefore, $\langle \xi, x - x_0 \rangle \leq 0$ for all $\xi \in \partial^\flat \varphi(x_0)$, as required. □

**Definition 2.** The function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be topological pseudoconvex at $x_0$ if

$$\varphi(x) < \varphi(x_0) \Longrightarrow \langle \xi, x - x_0 \rangle < 0 \qquad \forall \xi \in \partial^\flat \varphi(x_0).$$

A sufficient condition for the topological pseudoconvexity of quasiconvex functions is presented in following theorem.

**Theorem 6.** Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be an upper-semicontinuous quasiconvex function with $0_n \notin \partial^\flat \varphi(x_0)$ for some $x_0 \in \mathbb{R}^n$. Then $\varphi$ is topological pseudoconvex at $x_0$.

*Proof.* Assume that

$$\varphi(x) < \varphi(x_0). \tag{9}$$

Let $\xi$ be an arbitrary element in $\partial^\flat \varphi(x_0)$. We claim that

$$\Big\langle \xi, x - x_0 \Big\rangle < 0. \tag{10}$$

Suppose on the contrary that $\Big\langle \xi, x - x_0 \Big\rangle \geq 0$. Thus, Lemma 2 implies that $\Big\langle \xi, x - x_0 \Big\rangle = 0$. This means that there exists a sequence $\{u_l\}_{l=1}^\infty$ such that

$$\lim_{l\to\infty} u_l = x - x_0, \qquad \text{and} \qquad \Big\langle \xi, u_l \Big\rangle > 0, \quad \forall l \in \mathbb{N}.$$

The latter can be rewritten as

$$\left\langle \xi, (u_l + x_0) - x_0 \right\rangle > 0, \quad \forall l \in \mathbb{N}.$$

From this and Lemma 2, we conclude that $\varphi(u_l + x_0) \geq \varphi(x_0)$. After passing to the limit as $l \to \infty$, we deduce that $\varphi(x) \geq \varphi(x_0)$ by the upper-semicontinuity of $\varphi$. This contradicts (9), and so, (10) holds. □

Now, we can state a sufficient optimality condition of KKT type.

**Theorem 7.** (KKT sufficient condition) Let $\hat{x} \in \Omega$. Assume that $\vartheta$ is topological pseudoconvex at $\hat{x}$, and there exist $\lambda_j \geq 0$ for $j \in J(\hat{x})$, such that

$$0_n \in \partial^\flat \vartheta(\hat{x}) + \sum_{j \in J(\hat{x})} \lambda_j \partial^\flat \psi_j(\hat{x}). \tag{11}$$

Then $\hat{x}$ is a global solution of (P).

*Proof.* Set $J(\hat{x}) = \{j_1, \ldots, j_p\}$. According to (11), we can find some $\xi \in \partial^\flat \vartheta(\hat{x})$ and $\xi_{j_k} \in \partial^\flat \psi_{j_k}(\hat{x})$, $k = 1, 2, \ldots, p$, such that

$$\xi + \lambda_{j_1} \xi_{j_1} + \cdots + \lambda_{j_p} \xi_{j_p} = 0_n. \tag{12}$$

Suppose on the contrary that there exists some $x^* \in \Omega$ such that $\vartheta(x^*) < \vartheta(\hat{x})$. From $\xi \in \partial^\flat \vartheta(\hat{x})$ and the topological pseudoconvexity of $\vartheta$ at $\hat{x}$, we get

$$\langle \xi, x^* - \hat{x} \rangle < 0. \tag{13}$$

Since $\psi_{j_k}(x^*) \leq 0 = \psi_{j_k}(\hat{x})$ for $k = 1, 2, \ldots, p$, Lemma 2 allows us to conclude that

$$\left\langle \lambda_{j_k} \xi_{j_k}, x^* - \hat{x} \right\rangle \leq 0 \qquad \forall k = 1, 2, \ldots, p. \tag{14}$$

Owing to the inequalities (13) and (14), we deduce that

$$\left\langle \xi + \lambda_{j_1} \xi_{j_1} + \cdots + \lambda_{j_p} \xi_{j_p}, x^* - \hat{x} \right\rangle < 0,$$

which contradicts (12). The proof is complete. □

Now, we apply the optimality conditions which were presented in the previous theorems to give the weak and strong duality results for (P). We consider the following Mond-Weir [8] dual problem to (P):

$$\text{(MD):} \qquad \max \vartheta(y) \qquad \text{s.t.} \quad y \in \Lambda,$$

where $\Lambda$ is defined as:

$$\Lambda := \left\{ y \in \mathbb{R}^n \mid \exists \lambda_j \geq 0 \text{ for } j \in J(y), \text{ such that } \ 0_n \in \partial^\flat \vartheta(y) + \sum_{j \in J(y)} \partial^\flat \lambda_j \psi_j(y) \right\}.$$

**Theorem 8.** *(weak duality)* Suppose that $\hat{x} \in \Omega$ and $\hat{y} \in \Lambda$. If $\vartheta$ is topological pseudoconvex at $\hat{y}$, then $\vartheta(\hat{x}) \geq \vartheta(\hat{y})$.

*Proof.* Definition of $\Lambda$ and the assumption $\hat{y} \in \Lambda$ imply that there exist some $\zeta \in \partial^\flat \vartheta(\hat{y})$, $\lambda_j \geq 0$ and $\zeta_j \in \partial^\flat \psi_j(\hat{y})$ for $j \in J(\hat{y})$, such that

$$\zeta + \sum_{j \in J(\hat{y})} \lambda_j \zeta_j = 0_n. \tag{15}$$

Assume on the contrary that $\vartheta(\hat{x}) < \vartheta(\hat{y})$. So, the topological pseudoconvexity of $\vartheta$ at $\hat{x}$ implies that

$$\langle \zeta, \hat{x} - \hat{y} \rangle < 0. \tag{16}$$

On the other hand, the quasiconvexity of $\psi_j$s at $\hat{y}$, Lemma 2, and the inequality $\psi_j(\hat{x}) \leq 0 = \psi_j(\hat{y})$ for $j \in J(\hat{y})$ imply that

$$\langle \zeta_j, \hat{x} - \hat{y} \rangle \leq 0, \qquad \text{for all} \quad j \in J(\hat{y}).$$

The above inequality, (16), and $\lambda_j \geq 0$ for $j \in J(\hat{y})$ conclude that

$$\langle \zeta + \sum_{j \in J(\hat{y})} \lambda_j \zeta_j, \hat{x} - \hat{y} \rangle = \langle \zeta, \hat{x} - \hat{y} \rangle + \sum_{j \in J(\hat{y})} \lambda_j \langle \zeta_j, \hat{x} - \hat{y} \rangle < 0,$$

which contradicts (15). The proof is complete. □

**Theorem 9.** *(strong duality)* Let $\hat{x}$ and $\hat{y}$ be respectively the optimal solutions of (P) and (MD). Assume that $\vartheta$ and $\psi_j$ are regular at $\hat{x}$ as $j \in J(\hat{x})$, $\mathcal{I}(\Omega, \hat{x}) \neq \emptyset$, and the (GKTCQ) is satisfied at $\hat{x}$. If $\vartheta$ is topological pseudoconvex at $\hat{y}$ and $conv\big(\mathcal{Z}(\hat{x})\big)$ is a closed set, the optimal values of (P) and (MD) are equal, i.e.,

$$\vartheta(\hat{x}) = \vartheta(\hat{y}).$$

*Proof.* Since the hypothesis of Theorem 5 are all true, we can find some scalars $\lambda_j \geq 0,\ j \in J(\hat{x})$ such that

$$0_n \in \partial^\flat \vartheta(\hat{x}) + \sum_{j \in J(\hat{x})} \partial^\flat \lambda_j \psi_j(\hat{x}).$$

This implies that $\hat{x} \in \Lambda$, i.e., $\hat{x}$ is a feasible point for (MD). Because of $\hat{y}$ is optimal solution of (MD), the inequality $\vartheta(\hat{x}) \leq \vartheta(\hat{y})$ holds. This inequality and weak duality Theorem 9 imply $\vartheta(\hat{x}) = \vartheta(\hat{y})$, as required. □

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