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Research Article

On Edge Fuzzy Line Graphs and their Fuzzy Congraphs

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Abstract. In this paper, we introduce some new concepts of fuzzy graphs with the notion of degree of an edge in fuzzy line graphs and congraphs. Also, some properties and some lemmas of edge fuzzy line graphs and congraphs are studied. Finally, we state and prove some results related to these concepts.

Keywords. Fuzzy graph, Fuzzy line graph, Fully connected graph, Common neighborhood graph, Congraph.

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1 Introduction

In 1965, L.A. Zadeh introduced fuzzy sets [9]. Initially introduced by Kauffman in 1973, the notion of fuzzy graph was mainly developed by the papers written by Rosenfeld [8], who presented presented basic structural and connectivity concepts. On the other hand, Yeh and Bang developed the idea of different connectivity parameters and discussed their application. According to Rosenfeld, the fuzzy analogues of several there are graph-theoretic concepts, such as bridges, paths, cycles, trees, and connectedness. There are several theoretical developments in the theory of fuzzy graphs which are based on Rosenfeld's initial findings. Nowadays, fuzzy graph theory is applied in many different areas.

The concept of edge regular fuzzy graph was introduced in [6]. In this paper, we discuss the edge regular property of fuzzy line graphs. Some basic definitions in the next section are from [1, 2, 3, 4, 5, 7].

Let V be a non-empty finite set and $E \subseteq \{\{x,y\}; x,y \in V\}$. A fuzzy graph $G(V,E,\sigma,\mu)$ is defined by functions: $\sigma : V \longrightarrow (0,1]$, $\mu : E \longrightarrow (0,1]$ and $\mu\{x,y\} \leq \sigma(x) \wedge \sigma(y)$ for all $x,y \in V$. Let $|V(G)| = p$ and $|E(G)| = q$, then we say that $G(V,E,\sigma,\mu) = G(\sigma,\mu)$ is a (p,q)-fuzzy graph. Henceforth, the edge connecting the vertices x and y is denoted by $xy = \{x,y\}$. The notions $o(G) = \sum_{x \in V} \sigma(x)$ and $S(G) = \sum_{xy \in E} \mu(xy)$ are order and size of a $G(\sigma,\mu)$. Also, $G(\sigma,\mu)$ is strong, if $\mu(xy) = \sigma(x) \wedge \sigma(y)$ for all $xy \in E$ and $G(\sigma,\mu)$ is complete, if $\mu(xy) = \sigma(x) \wedge \sigma(y)$ for all $x,y \in V$.

In $G(\sigma,\mu)$, if $\sigma(x) = \mu(xy) = 1$ for every $x,y \in V$, then this graph is called the underlying crisp graph and is denoted by $G(V,E)$. In other words, in the fuzzy graph $G(\sigma,\mu)$, if for every $v \in V$ we have $\sigma(v) = 1$ and for every $e \in E$ have $\mu(e) = 1$, then G is a crisp graph. This shows that every crisp graph is a fuzzy graph.

Let $G(\sigma,\mu)$ be a fuzzy graph, then the degree of a vertex x is $d_G(x) = \sum_{x \neq y} \mu(xy)$. If for every $v \in V(G)$, v has same degree k, then G is a regular fuzzy graph or $k-$ regular fuzzy graph.

Let $V(G) = \{v_1, v_2, \ldots, v_p\}$ and $E(G) = \{e_1, e_2, \ldots, e_q\}$. The $p \times p$ matrix $A = A(G)$ whose (i,j) entry is defined by $a_{ij} = \mu(v_i v_j)$ is called the *adjacency matrix* of G. Also, the $p \times p$ matrix D_G whose (i,j) entry is defined by

$$d_{ij} = \begin{cases} d_G(v_i), & i = j, \\ 0, & \text{o.w.,} \end{cases}$$

is called the *degree matrix* of G.

A $p \times q$ matrix M, with rows indicanting the vertices and columns indicanting the edges, and (i,j) entry is defined by

$$m_{ij} = \begin{cases} \mu(v_i v_t), & \text{if } v_i \text{ is an endpoint of edge } e_j = v_i v_t, \\ 0, & \text{o.w.,} \end{cases}$$

and is called (vertex-edge) incidence matrix of G.

The Fuzzy degree matrix of G is the matrix D_F with order $p \times p$ is defined by

$$c_{ij} = \begin{cases} \sum\limits_{v_i \neq v_k} \mu^2(v_i v_k), & i = j, \\ 0, & \text{o.w.}. \end{cases}$$

The matrix E with order $q \times q$ and with entries

$$e_{ij} = \begin{cases} \mu(e_i), & i = j, \\ 0, & \text{o.w.}. \end{cases}$$

is called the edge matrix of G.

2 Common Neighbourhood Graph

Definition 1. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices of size $n \times n$. Then, we define $C = A \odot B$ as the $n \times n$ matrix whose (i, j) entry is $a_{ij}.b_{ij}$.

Lemma 1. Let $G(\sigma, \mu)$ be a (p, q)-fuzzy graph, such that A, M and D_F are the adjacency matrix, incidence and fuzzy degree graph of G respectively. Then

$$M.M^T = A \odot A + D_F.$$

Proof. Let $A = [a_{ij}]_{p\times p}$, $M = [m_{ij}]_{p\times q}$, $D_F = [c_{ij}]_{p\times p}$ and let $A \odot A = [t_{ij}]_{p\times p}$ and $MM^T = [b_{ij}]_{p\times p}$, then for $i \neq j$ we get

$$\begin{aligned} b_{ij} &= \sum_{k=1}^{q} m_{ik}.m^T_{kj} \\ &= \sum_{k=1}^{q} m_{ik} m_{jk} = \mu^2(e_k) = \mu^2(v_i v_j). \end{aligned}$$

For $m_{ik} m_{jk} \neq 0$, we have $m_{ik} \neq 0$ and $m_{jk} \neq 0$, the vertex v_i is an endpoint of edge e_k and vertex v_j is an endpoint of edge e_k. Hence $e_k = v_i v_j$. Therefore, $b_{ij} = \mu^2(v_i v_j) = a_{ij}.a_{ij} + c_{ij} = t_{ij} + 0$. Thus $M.M^T = A \odot A + D_F$.

If $i = j$ and v_i is an endpoint of edge $e_k = v_i v_t$, then

$$\begin{aligned} b_{ii} &= \sum_{k=1}^{q} m_{ik}.m^T_{ki} = \sum_{k=1}^{q} m_{ik} m_{ik} \\ &= \sum_{k=1}^{q} m^2_{ik} = \sum_{v_i \neq v_t} \mu^2(v_i v_t) = c_{ii}. \end{aligned}$$

Therefore, $b_{ii} = a_{ii}.a_{ii} + c_{ii} = 0 + c_{ii}$. Also, in this case, we get $M.M^T = A \odot A + D_F$. □

We can obtain similar results for crisp graphs. If G is a crisp graph then, $D_F = D$, $A \odot A = A$. As a consequence of Lemma 1, we have the following remarks for crisp graphs.

Remark 1. Let $G(V,E)$ be a (p,q)-graph and let A, M and D be the adjacency, incidence and degree matrix graph G respectively. Then

$$M.M^T = A + D.$$

Let $G(V,E)$ be a graph. The *common neighbourhood graph* (or shorter, *congraph*) of G, denoted by $con(G)$, is the graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ in which two vertices are adjacent if and only if they have a common neighbour in G. In other words, for every $x, y \in V(G)$,

$$xy \in E(con(G)) \Longleftrightarrow N_G(x) \cap N_G(y) \neq \emptyset.$$

The fuzzy congraph of $G(\sigma,\mu)$ is $con(G)(\omega,\lambda)$ such that $\omega(x) = \sigma(x)$ and $\lambda(uv) = \min_{x \in H}\{\mu(ux).\mu(vx)\}$, where $H = N_G(u) \cap N_G(v)$.

Lemma 2. Let $G(\sigma,\mu)$ be a (p,q)-fuzzy graph and $con(G)(\omega,\lambda)$ be the fuzzy congraph of $G(\sigma,\mu)$. If G has no cycle of size 4, then

$$d_{con(G)}(v) = \sum_{u \neq v} \mu(vu).d_G(u) - \sum_{u \neq v} \mu^2(vu).$$

Proof.

$$\begin{aligned} d_{con(G)}(v) &= \sum_{u \neq v} \lambda(vu) \\ &= \sum_{u \neq v} \min(\mu(vw).\mu(wu)), \end{aligned}$$

where $w \in H = N_G(v) \cap N_G(u)$. Since G has no cycle of size 4, hence $H = \{w\}$. Therefore,

$$\begin{aligned} d_{con(G)}(v) &= \sum_{u \neq v} \min\left(\mu(vw).\mu(wu)\right) \\ &= \sum_{vw, uw \in E(G^*)} \mu(vw)\,\mu(wu) \\ &= \sum_{vw \in E(G^*)} \mu(vw) \sum_{u \neq w} \mu(uw) - \sum_{w \neq v} \mu^2(vw) \\ &= \sum_{w \neq v} \mu(vw).d_G(w) - \sum_{w \neq v} \mu^2(vw) \\ &= \sum_{u \neq v} \mu(vu).d_G(u) - \sum_{u \neq v} \mu^2(vu). \end{aligned}$$

□

In particular, let G be a graph. Then we have the following remark.

Remark 2. Let $G(V,E)$ be a (p,q)-graph and $con(G)$ be a (p,q')-graph. If G has no cycle of size 4, then for every $v \in V(G)$ we have

$$d_{con(G)}(v) = \sum_{u \in N_G(v)} d_G(u) - d_G(v).$$

Definition 2. [5] Let $G(\sigma,\mu)$ be a (p,q)-fuzzy graph, then we define:

(۱) $M_1(G) = \sum_{v_i\in V} d_G^2 v_i$,

(۲) $M_2(G) = \sum_{v_iv_j\in E} \mu(v_iv_j) d_G v_i . d_G v_j$,

(۳) $F(G) = \sum_{v_i\in V} d_G^3 v_i$.

Lemma 3. Let $G(\sigma,\mu)$ be a (p,q)-fuzzy graph. Then:

$$\sum_{v_iv_j\in E} \mu(v_iv_j)(d_G^k v_i + d_G^k v_j) = \sum_{v_i\in V} d_G^{k+1} v_i.$$

In particular,

$$\sum_{v_iv_j\in E} \mu(v_iv_j)(d_G v_i + d_G v_j) = \sum_{v_i\in V} d_G^2 v_i = M_1(G),$$

and

$$\sum_{v_iv_j\in E} \mu(v_iv_j)(d_G^2 v_i + d_G^2 v_j) = \sum_{v_i\in V} d_G^3 v_i = F(G).$$

Proof.

$$\begin{aligned}
\sum_{v_iv_j\in E} \mu(v_iv_j)(d_G^k v_i + d_G^k v_j) &= \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p} \mu(v_iv_j)(d_G^k v_i + d_G^k v_j) \\
&= \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p} \mu(v_iv_j) d_G^k v_i + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p} \mu(v_iv_j) d_G^k v_j \\
&= \frac{1}{2}\sum_{i=1}^{p} d_G^k v_i \sum_{j=1}^{p} \mu(v_iv_j) + \frac{1}{2}\sum_{j=1}^{p} d_G^k v_j \sum_{i=1}^{p} \mu(v_iv_j) \\
&= \frac{1}{2}\sum_{i=1}^{p} d_G^k v_i\, d_G(v_i) + \frac{1}{2}\sum_{j=1}^{p} d_G^k v_j\, d_G(v_j) \\
&= \sum_{v_i\in V} d_G^{k+1} v_i
\end{aligned}$$

□

Let $G(V,E)$ be a graph. The line graph of G, denoted by $L(G)(Z,W)$, is a graph such that $Z = E$.

For any pair $\{e_1,e_2\}$ of distinct edges in G, they are adjacent (as vertices) in the line graph if and only if they are adjacent (as edges) in the original graph which means $\{e_1,e_2\} \in W$ if and only if $|e_1 \cap e_2| = 1$. The graph $L(G)(Z,W)$ is called the line graph of G.

3 Fuzzy Line Graph

The fuzzy line graph of $G(\sigma,\mu)$ is $L(G)(\omega,\lambda)$ such that for every $e = uv \in E$ we have $\omega(e) = \mu(uv)$ and for every $e_1 = uv_1,\ \ e_2 = uv_2$ we have $\lambda(e_1e_2) = \mu(uv_1).\mu(uv_2) = \omega(e_1).\omega(e_2)$.

Lemma 4. Let $G(\sigma,\mu)$ be a (p,q)-fuzzy graph and $L(G)(\omega,\lambda)$ be (q,q')-fuzzy line graph. Then

$$d_{L(G)}(e) = \mu(v_iv_j)\Big(d_G(v_i) + d_G(v_j) - 2\mu(v_iv_j)\Big).$$

Proof. For every $e = v_iv_j \in E$, $t \neq j$ and $s \neq i$, we set $e' = v_sv_t$, then we have:

$$\begin{aligned}
d_{L(G)}(e) &= \sum_{e'\neq e}\lambda(ee') \\
&= \sum_{e\neq e'=v_iv_t\in E}\lambda(ee') + \sum_{e\neq e'=v_jv_s\in E}\lambda(ee') \\
&= \sum_{v_t\neq v_i,\quad t\neq j}\mu(v_iv_j)\,\mu(v_iv_t) + \sum_{v_s\neq v_j,\quad s\neq i}\mu(v_iv_j)\,\mu(v_jv_s) \\
&= \mu(v_iv_j)\sum_{v_t\neq v_i,\quad t\neq j}\mu(v_iv_t) + \mu(v_iv_j)\sum_{v_s\neq v_j,\quad s\neq i}\mu(v_iv_j) \\
&= \mu(v_iv_j)(\sum_{v_t\neq v_i}\mu(v_iv_t) - \mu(v_iv_j)) + \mu(v_iv_j)(\sum_{v_s\neq v_j}\mu(v_jv_s) - \mu(v_iv_j)) \\
&= \mu(v_iv_j)[d_G(v_i) + d_G(v_j) - 2\mu(v_iv_j)]
\end{aligned}$$

□

Remark 3. Let $G(V,E)$ be a graph and $L(G)(Z,W)$ be the line graph of G. Then for every $e = uv \in E(G)$,

$$d_{L(G)}(e) = d_G(u) + d_G(v) - 2.$$

Lemma 5. Let $G(\sigma,\mu)$ be a (p,q)-fuzzy graph, such that M and E are the incidence matrix and edge of matrix G and L, respectively. Let L be the adjacency matrix of the line graph G. Then

$$M^T.M = L + 2E \odot E$$

Proof. Let $M = [m_{ij}]_{p\times q}$, $L = [l_{ij}]_{q\times q}$, $E = [e_{ij}]_{q\times q}$ and $M^TM = [b_{ij}]_{q\times q}$. Also, if v_k is an endpoint of edge $e_i = v_kv_t$ and if v_k is an endpoint of edge $e_j = v_kv_s$, then for $i \neq j$ we get

$$\begin{aligned}
b_{ij} &= \sum_{k=1}^{p} m^T_{ik}.m_{kj} \\
&= \sum_{k=1}^{p} m_{ki}m_{kj} = \mu(e_i)\mu(e_j) = \mu(v_kv_t)\mu(v_kv_s).
\end{aligned}$$

For $m_{ki}m_{kj} \neq 0$, we have $m_{ki} \neq 0$ and $m_{kj} \neq 0$. The vertex v_k is an endpoint of edge e_i and vertex v_k is an endpoint of edge e_j. This is implies that $\{e_i, e_j\}$ is an edge in line graph G. That is, $e_ie_j \in E_{L(G)}$, where $e_i = v_kv_s$ and $e_j = v_kv_t$. Therefore, $b_{ij} = \mu(v_kv_s)\mu(v_kv_t) = \lambda(e_ie_j) = l_{ij} + 0 = l_{ij} + e_{ij}$. Thus $M^T.M = L + 2E \odot E$.

If $i = j$, then for $e_i = v_tv_s$ we have:

$$\begin{aligned} b_{ii} &= \sum_{k=1}^{p} m_{ki}^T.m_{kj} \\ &= \sum_{k=1}^{p} m_{ki}m_{ki} = \sum_{k=1}^{p} m_{ki}^2 = m_{ts}^2 + m_{st}^2 \\ &= \mu^2(v_tv_s) + \mu^2(v_sv_t) = 2\mu^2(e_i) = 2e_{ii}.e_{ii}. \end{aligned}$$

Therefore, $b_{ii} = 2\mu^2(e_i) = l_{ii} + 2e_{ii}.e_{ii} = 0 + 2e_{ii}.e_{ii}$. Also, in this case, we get

$$M^T.M = L + 2E \odot E.$$

□

Remark 4. Let G be a (p, q)- graph, such that M and L are the incidence matrix graph of G and the adjacency matrix line graph of G respectively. Then

$$M^T.M = L + 2I_{q\times q}$$

Lemma 6. Let $G(\sigma, \mu)$ be a (p, q)-fuzzy graph and $L(G)(\omega, \lambda)$ be (q, q')-fuzzy line graph. Then

$$2q' = M_1(G) - 2\sum_{k=1}^{q} \mu^2(e_k).$$

Proof.

$$\begin{aligned} 2q' &= \sum_{k=1}^{q} d_{L(G)}(e_k) = \sum_{e_k = v_{ik}v_{jk}} d_{L(G)}(e_k) \\ &= \sum_{v_{ik}v_{jk} \in E(G)} d_{L(G)}(e_k) \\ &= \sum_{v_{ik}v_{jk} \in E(G)} \mu(v_{ik}v_{jk})(d_G(v_{ik}) + d_G(v_{jk})) - 2\mu(v_{ik}v_{jk}) \\ &= \sum_{v_{ik}v_{jk} \in E(G)} \mu(v_{ik}v_{jk})(d_G(v_{ik}) + d_G(v_{jk})) - 2\sum_{k=1}^{q} \mu^2(e_k) \\ &= M_1(G) - 2\sum_{k=1}^{q} \mu^2(e_k). \end{aligned}$$

□

Remark 5. Let $G(V,E)$ be a (p,q)-graph and $L(G)(q,q')$ be the line graph of G. Then

$$q' = \frac{1}{2}(M_1(G) - 2q).$$

Lemma 7. Let $G(\sigma,\mu)$ be a (p,q)-fuzzy graph, such that A and B are the adjacency matrix graphs of G and $con(G)$ respectively.

If G has no cycle of size 4, then $A^2 = B + D_F$ where D_F is the fuzzy degree matrix of G.

Proof. Let $A = [a_{ij}]_{p\times p}$ and $B = [b_{ij}]_{p\times p}$. For $i \neq j$ we get

$$\begin{aligned} a_{ij}^{(2)} &= \sum_{k=1}^{p} a_{ik}.a_{kj} \\ &= \mu(v_i v_s)\mu(v_s v_j). \end{aligned}$$

For $a_{ki}a_{kj} \neq 0$, we have $a_{ki} \neq 0$ and $a_{kj} \neq 0$.

The vertex v_i is adjacent to the vertex v_k. Also, the vertex v_j is adjacent to the vertex v_k. That is $\{v_i v_j\}$ is an edge in graph $con(G)$. Since G has no cycle of size 4, then there exists only vertex $v_s \in V$ such that $v_i v_s \;\; and \;\; v_s v_j \in E(G)$. Therefore, $a_{ij}^{(2)} = \mu(v_i v_s)\mu(v_s v_j) = b_{ij} + c_{ij} = b_{ij} + 0$. In this case we get $A^2 = B + D_F$.

If $i = j$, then

$$\begin{aligned} a_{ii}^{(2)} &= \sum_{k=1}^{p} a_{ik}.a_{ki} \\ &= \sum_{k=1}^{p} a_{ik}^2 = \sum_{v_k \neq v_i} \mu^2(v_i v_k). \end{aligned}$$

Therefore, $a_{ii}^{(2)} = \sum_{v_k \neq v_i} \mu^2(v_i v_k) = b_{ii} + c_{ii} = 0 + c_{ii}$. Also, in this case we get $A^2 = B + D_F$.

□

Remark 6. Let $G(V,E)$ be a (p,q)-graph, such that A and B are the adjacency matrix graphs of G and $con(G)$ respectively. If G has no cycle of size 4, then $A^2 = B + D$ where D is the degree matrix of G.

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References

[1] Nagoor Gani A., Malarvizhi J., 2008. "Isomorphism on fuzzy graphs", International Journal of Computational and Mathematical Sciences, 2(4), 190-196.

[2] Nagoor Gani A., Malarvizhi J., 2009. "Properties of complement of a fuzzy graph", International Journal of Algorithms, Computing and Mathematics, 2(3), 73- 83.

[3] Nagoor Gani A., Radha K., 2008. "On regular fuzzy graphs", Journal of Physical Sciences, 12, 33-40.

[4] Nusantara T., Rahmadani D., Asmianto, Trisanti Y., Bin Gani A., 2021. "Anti fuzzy line graphs", Journal of Physics: Conference Series, 1783, 012097, doi:10.1088/1742-6596/1783/1/012097

[5] Radha K., Kumaravel N., 2014. "On edge regular fuzzy graphs", International Journal of Mathematical Archive, 5(9), 100-112.

[6] Radha K., Kumaravel N., 2014. "Some properties of edge regular fuzzy graphs", Jamal Academic Research Journal (JARJ), Special issue, 121-127.

[7] Radha K., Kumaravel N., 2016. " On edge regular fuzzy line graphs", International Journal of Computational and Applied Mathematics, 11(2), 105-118.

[8] Rosenfeld A., 1975. "Fuzzy graphs", in: L.A. Zadeh, K.S. Fu and M. Shimura, Eds., Fuzzy Sets and Their Applications, Academic Press, New York, 77 - 95.

[9] Zadeh L.A., 1965. "Fuzzy sets", Information and Control, 8, 338-353.

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