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Research Article

Guignard Qualifications and Stationary Conditions for Mathematical Programming with Nonsmooth Switching Constraints

Fatemeh Gorgini Shabankareh[1], **Nader Kanzi**[2], **Javad Izadi**[3*], **Kamal Fallahi**[4]

[1,2,3,4]Department of Mathematics, Payame Noor University (PNU),
P.O. Box. 19395-4697, Tehran, Iran.

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Abstract. In this paper, some constraint qualifications of the Guignard type are defined for optimization problems with continuously differentiable objective functions and locally Lipschitz switching constraints. Then, a new type of stationary condition, named as parametric stationary condition, is presented for the problem, and it is shown that all the stationarity conditions in various papers can be deduced from it. This paper can be considered as an extension of a recent article (see Kanzow, et al.) to the nonsmooth case. Finally, the article ends with two important examples. The results of the article are formulated according to Clark subdifferential and using nonsmooth analysis methods.

Keywords. Constraint qualification, Stationary conditions, Optimality conditions, Switching constraints.

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* Corresponding author
f.gorgini2016@gmail.com, nad.kanzi@gmail.com, j.izadi@pnu.ac.ir, fallahi1361@gmail.com
http://mathco.journals.pnu.ac.ir

1 Introduction

This paper focuses on the following "mathematical programming with switching constraints" (MPSC, in brief):

$$(P) \qquad \min \quad f(x)$$
$$s.t. \quad G_i(x)H_i(x) = 0, \quad i = 1, \ldots, m,$$
$$x \in \mathbb{R}^n$$

where, $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable, and $H_i, G_i : \mathbb{R}^n \to \mathbb{R}$ are locally Lipschitz for all $i \in I := \{1, \ldots, m\}$.

Recently, MPSCs have been introduced by Mehlitz [9] as a new type of optimization problem. Mehlitz showed that some of the well-known constraint qualifications such as Mangasarian-Fromovitz and linear independent constraint qualifications (CQ) do not hold at each feasible point of MPSCs, and he introduced Abadie and Guignard type CQs for these problems. Then, he presented some different optimality conditions, named stationary conditions, for MPSCs under these CQs. Due to the wide applications of MPSC in control theory, physics, topological optimization, etc., research on it was considered by many researchers. The exact penalty method, the relaxation schemes, and the topological approach for solving MPSCs are studied in [8], [5] and [12], respectively.

In the previous works that referenced earlier, all functions which define MPSC are continuously differentiable. To the best of our knowledge, the current article is the first that studies the stationary conditions for MPSCs where their constraints are non-smooth, meaning that they are not necessarily differentiable. In this paper, we assume that the objective function is differentiable.

The structure of subsequent sections of this paper is as follows. Section 2 contains the elementary definitions, notations, theorems, and relations that are required in the sequel. In Section 3, we will introduce some Guignard type CQs for problem (P). Also, different kinds of stationary conditions at optimal solutions of (P) are presented in this section.

2 Notations and Preliminaries

In this section, we recall some definitions and theorems in non-smooth analysis, which are widely used in the sequel, from [3, 4].

Given a nonempty set $B \subseteq \mathbb{R}^n$, we denote by $\overline{B}$ and $cone(B)$, the closure of B and the convex cone of B, respectively; i.e., $cone(B) = conv(\bigcup_{\beta \geq 0} \beta B)$ where $conv(A)$ denotes the convex hull of A. The polar cone of B is defined by

$$B^0 := \{x \in \mathbb{R}^n \mid \langle x, b \rangle \leq 0, \quad \forall b \in B\},$$

where $\langle x, b \rangle$ refers to the standard inner product of x and b in $\mathbb{R}^n$. Also, the orthogonal cone to B is denoted by $B^{\perp}$,

$$B^{\perp} := B^0 \cap (-B)^0 = \{x \in \mathbb{R}^n \mid \langle x, b\rangle = 0, \quad \forall b \in B\}.$$

Let $\emptyset^0 = \emptyset^{\perp} := \mathbb{R}^n$. The bipolar theorem [4] states that

$$B^{00} := (B^0)^0 = \overline{cone}(B) := \overline{cone(B)}. \tag{1}$$

Clearly, the following relations hold,

$$B^0 = \Big(cone(B)\Big)^0 = \Big(\overline{cone}(B)\Big)^0, \tag{2}$$

$$B_1 \subseteq B_2 \implies B_2^0 \subseteq B_1^0. \tag{3}$$

Suppose that $B_1, \dots, B_p$ are convex subsets of $\mathbb{R}^n$, and $\mathcal{B} := \bigcup_{\ell=1}^{p} B_\ell$. Then, it is easy to see that [4]

$$cone(\mathcal{B}) = \Big\{ \sum_{\ell=1}^{p} \beta_\ell b_\ell \mid b_\ell \in B_\ell,\ \beta_\ell \geq 0 \Big\}. \tag{4}$$

The Buligand tangent cone (or contingent cone) and the Frechet normal cone of a nonempty set $B \subseteq \mathbb{R}^n$ at $x_0 \in B$ are respectively denoted by $\Gamma(B, x_0)$ and $N(B, x_0)$, defined as

$$\Gamma(B, x_0) := \Big\{ v \in \mathbb{R}^n \mid \exists t_\ell \downarrow 0,\ \exists v_\ell \to v \text{ such that } x_0 + t_\ell v_\ell \in B \quad \forall \ell \in \mathbb{N} \Big\},$$

$$N(B, x_0) := \Big(\Gamma(B, x_0)\Big)^0.$$

Theorem 1. [4] Let $x_0 \in B \subseteq \mathbb{R}^n$ be a minimizer of a continuously differentiable function $\varphi : \mathbb{R}^n \to \mathbb{R}$ on B. Then,

$$0_n \in \{\nabla\varphi(x_0)\} + N(B, x_0),$$

in which $0_n := (\overbrace{0, \dots, 0}^{n\ times})$ is the zero vector of $\mathbb{R}^n$, and $\nabla\varphi(x_0)$ denotes the classic gradient of φ at x_0.

A real-valued function $\psi : \mathbb{R}^n \to \mathbb{R}$ is said to be Lipschitz near $x_0 \in \mathbb{R}^n$ if there exist a neighborhood U_0 for x_0 and a positive number $L_0 > 0$ such that

$$|\psi(x) - \psi(x_0)| \leq L_0 \|x - x_0\|, \qquad \text{for all } x \in U_0.$$

ψ is said to be locally Lipschitz when it is Lipschitz near all $x_0 \in \mathbb{R}^n$. For a given locally Lipschitz function $\psi : \mathbb{R}^n \to \mathbb{R}$, the Clarke directional derivative of ψ at x_0 in the direction $v \in \mathbb{R}^n$, and the Clarke subdifferential of ψ at x_0 are respectively defined by [3]

$$\psi^0(x_0; v) := \limsup_{y \to x_0,\ t \downarrow 0} \frac{\psi(y + tv) - \psi(y)}{t},$$

$$\partial_c \psi(x_0) := \Big\{ \xi \in \mathbb{R}^n \mid \langle \xi, v\rangle \leq \psi^0(x_0; v) \qquad \text{for all } v \in \mathbb{R}^n \Big\}.$$

The Clarke subdifferential is a generalization of the classical derivative; i.e., if ψ is continuously differentiable at x_0, we have $\partial_c \psi(x_0) = \{\nabla \psi(x_0)\}$. Also, the Clarke subdifferential of each locally Lipschitz function at all points in its domain is always nonempty, convex, and compact.

As the last point of this section, we recall ([4]) that if φ and ψ_t, for $t = 1, \dots, p$, are locally Lipschitz functions from $\mathbb{R}^n$ to $\mathbb{R}$, we say that the optimization problem

$$\min \ \varphi(x) \ \text{ s.t. } x \in \Pi := \Big\{ x \in \mathbb{R}^n \mid \psi_t(x) \le 0, \quad t = 1, 2, \dots p \Big\},$$

satisfies the Guignard constraint qualification at $x_0 \in \Pi$ if

$$\Big(\bigcup_{t \in T_0} \partial_c \psi_t(x_0) \Big)^{\circ} \subseteq \overline{cone}\Big(\Gamma(\Pi, x_0) \Big),$$

where $T_0 := \{t \mid \psi_t(x_0) = 0\}$.

3 The Main Results

Throughout this article, we suppose that the feasible set of (P), named S, is nonempty, i.e.,

$$S := \{x \in \mathbb{R}^n \mid G_i(x) H_i(x) = 0, \quad i \in I\} \neq \emptyset.$$

Let $\hat{x} \in S$ be a feasible point that will be fixed throughout this section. We divide the index set I into the following three index sets:

$$\begin{aligned} I_G &:= \{i \in I \mid H_i(\hat{x}) \neq 0, \ G_i(\hat{x}) = 0\}, \\ I_H &:= \{i \in I \mid H_i(\hat{x}) = 0, \ G_i(\hat{x}) \neq 0\}, \\ I_{GH} &:= \{i \in I \mid H_i(\hat{x}) = 0, \ G_i(\hat{x}) = 0\}. \end{aligned}$$

Notice that, $I = I_G \cup I_H \cup I_{GH}$. As shown in [9], the problem

$$\begin{array}{llll} (P^*) & \min & f(x) & \\ & s.t. & G_i(x) = 0, & i \in I_G, \\ & & H_i(x) = 0, & i \in I_H, \end{array}$$

is locally equivalent to (P) when $I_{GH} = \emptyset$. In order to add the constraints related to index I_{GH} to (P^*), we consider two index sets $I_1 \subseteq I_{GH}$ and $I_2 \subseteq I_{GH}$ in such a way that $I_1 \cup I_2 = I_{GH}$, then, we add the constraints corresponding to I_1 and I_2 to the first and second lines of the constraints of problem (P^*), respectively. So, we obtain the following parametric problem which is defined in terms of parameters I_1 and I_2:

$$\begin{array}{llll} (P_{I_2}^{I_1}) & \min & f(x) & \\ & s.t. & G_i(x) = 0, & i \in I_G \cup I_1, \\ & & H_i(x) = 0, & i \in I_H \cup I_2. \end{array}$$

Remark 1. The index sets I_1 and I_2 can be such that $I_1 \cap I_2 \neq \emptyset$, and they do not need to be separate. Also, one of the index sets I_1 and I_2 may be equal to empty (in this case, the other is necessarily equal to I_{GH}).

For the sake of simplicity, motivated by [6, 7], for all $J_1 \subseteq I$ and $J_2 \subseteq I$, put

$$\mathcal{G}^{J_1} := \bigcup_{i\in J_1} \partial_c G_i(\hat{x}), \qquad \mathcal{H}^{J_2} := \bigcup_{i\in J_2} \partial_c H_i(\hat{x}),$$

$$\mathcal{Z}_{J_2}^{J_1} := \mathcal{G}^{J_1} \cup \Big(-\mathcal{G}^{J_1}\Big) \cup \mathcal{H}^{J_2} \cup \Big(-\mathcal{H}^{J_2}\Big).$$

Note that, corresponding to (P^*) and $(P_{I_2}^{I_1})$, we obtain the linearized cones $\Big(\mathcal{G}^{I_G}\Big)^{\perp} \cap \Big(\mathcal{H}^{I_H}\Big)^{\perp}$ and $\Big(\mathcal{G}^{I_G\cup I_1}\Big)^{\perp} \cap \Big(\mathcal{H}^{I_H\cup I_2}\Big)^{\perp}$, respectively. Since, the equalities

$$\Big(\mathcal{G}^{I_G}\Big)^{\perp} \cap \Big(\mathcal{H}^{I_H}\Big)^{\perp} = \Big(\mathcal{Z}_{I_H}^{I_G}\Big)^0, \quad \text{and} \quad \Big(\mathcal{G}^{I_G\cup I_1}\Big)^{\perp} \cap \Big(\mathcal{H}^{I_H\cup I_2}\Big)^{\perp} = \Big(\mathcal{Z}_{I_H\cup I_2}^{I_G\cup I_1}\Big)^0,$$

are clearly true, we are guided by the following definition.

Definition 1. We say that (P) satisfies the

(i): generalized Guignard constraint qualification, denoted by GGCQ, at $\hat{x}$, if

$$\Big(\mathcal{Z}_{I_H}^{I_G}\Big)^0 \subseteq \overline{cone}\Big(\Gamma(S,\hat{x})\Big).$$

(ii): (I_1, I_2)-parametric Guignard constraint qualification, denoted by PGCQ$_{I_2}^{I_1}$, at $\hat{x}$, if

$$\Big(\mathcal{Z}_{I_H\cup I_2}^{I_G\cup I_1}\Big)^0 \subseteq \overline{cone}\Big(\Gamma(S,\hat{x})\Big).$$

Remark 2. Since the feasible set of (P^*) contains S and the feasible set of $(P_{I_2}^{I_1})$ is contained in S, the following statements are true.

- Each optimal solution of (P) which is feasible for $(P_{I_2}^{I_1})$ is also an optimal solution for $(P_{I_2}^{I_1}))$.
- PGCQ$_{I_2}^{I_1}$ (resp. GGCQ) is not equivalent to Guignard CQ for $(P_{I_2}^{I_1})$ (resp. (P^*)).

Before stating the necessary optimality conditions for (P), we need to prove the following lemma.

Lemma 1. If J_1 and J_2 are two subsets of I, then

$$cone\Big(\mathcal{Z}_{J_2}^{J_1}\Big) = \bigcup_{(\mu_r)\in\mathbb{R}^{|J_1|}} \bigcup_{(\eta_s)\in\mathbb{R}^{|J_2|}} \Big(\sum_{r\in J_1} \mu_r \partial_c G_r(\hat{x}) + \sum_{s\in J_2} \eta_s \partial_c H_s(\hat{x})\Big).$$

Proof. Since the Clarke subdifferential of each locally Lipschitz function is convex, the definition of $\mathcal{Z}_{J_2}^{J_1}$ and (4) conclude that

$$cone\Big(\mathcal{Z}_{J_2}^{J_1}\Big) = \Big\{\sum_{r\in J_1}\alpha_r\hat{g}_r + \sum_{r\in J_1}\beta_r\tilde{g}_r + \sum_{s\in J_2}\gamma_s\hat{h}_s + \sum_{s\in J_2}\omega_s\tilde{h}_s \mid \alpha_r,\beta_r,\gamma_s,\omega_s \geq 0,$$

$$\hat{g}_r \in \partial_c G_r(\hat{x}),\ \tilde{g}_r \in \partial_c(-G_r)(\hat{x}),\ \hat{h}_s \in \partial_c H_s(\hat{x}),\ \tilde{h}_s \in \partial_c(-H_s)(\hat{x})\Big\}.$$

Owing to $\partial_c(-G_r)(\hat{x}) = -\partial_c G_r(\hat{x})$ as $r \in J_1$, for each $\tilde{g}_r \in \partial_c(-G_r)(\hat{x})$, there exists a $\hat{g}_r \in \partial_c G_r(\hat{x})$ such that $\tilde{g}_r = -\hat{g}_r$. Similarly, for each $\tilde{h}_s \in \partial_c(-H_s)(\hat{x})$, there exists a $\hat{h}_s \in \partial_c H_s(\hat{x})$ such that $\tilde{h}_s = -\hat{h}_s$ as $s \in J_2$. Thus,

$$cone\Big(\mathcal{Z}_{J_2}^{J_1}\Big) = \Big\{\sum_{r\in J_1}(\alpha_r - \beta_r)\hat{g}_r + \sum_{s\in J_2}(\gamma_s - \omega_s)\hat{h}_s \mid \alpha_r,\beta_r,\gamma_s,\omega_s \geq 0,$$

$$\hat{g}_r \in \partial_c G_r(\hat{x}),\ \hat{h}_s \in \partial_c H_s(\hat{x})\Big\}.$$

Taking $\mu_r := \alpha_r - \beta_r \in \mathbb{R}$ and $\eta_s := \gamma_s - \omega_s \in \mathbb{R}$ as $r \in J_1$ and $s \in J_2$, the latter equality implies that

$$\begin{aligned} cone\Big(\mathcal{Z}_{J_2}^{J_1}\Big) &= \Big\{\sum_{r\in J_1}\mu_r\hat{g}_r + \sum_{s\in J_2}\eta_s\hat{h}_s \mid \mu_r,\eta_s \in \mathbb{R},\ \hat{g}_r \in \partial_c G_r(\hat{x}),\ \hat{h}_s \in \partial_c H_s(\hat{x})\Big\} \\ &= \bigcup_{(\mu_r)\in\mathbb{R}^{|J_1|}}\ \bigcup_{(\eta_s)\in\mathbb{R}^{|J_2|}}\Big(\sum_{r\in J_1}\mu_r\partial_c G_r(\hat{x}) + \sum_{s\in J_2}\eta_s\partial_c H_s(\hat{x})\Big). \end{aligned}$$

□

Now, we state the nonsmooth version of strongly stationary (S-stationary, in short) condition presented in [9].

Theorem 2. (S-Stationary Condition): Suppose that $\hat{x}$ is an optimal solution of (P) and the GGCQ holds at $\hat{x}$. If $\overline{cone}\Big(\mathcal{Z}_{I_H}^{I_G}\Big) = cone\Big(\mathcal{Z}_{I_H}^{I_G}\Big)$ (equivalently, $cone\Big(\mathcal{Z}_{I_H}^{I_G}\Big)$ is closed), then there exist some multipliers $\lambda_i^G, \lambda_i^H \in \mathbb{R}$ as $i \in I$, such that

$$-\nabla f(\hat{x}) \in \sum_{i\in I}\Big(\lambda_i^G\partial_c G_i(\hat{x}) + \lambda_i^H\partial_c H_i(\hat{x})\Big), \tag{5}$$

$$\lambda_i^G = 0, \quad i \in I_H \cup I_{GH}, \quad \text{and} \quad \lambda_i^H = 0, \quad i \in I_G \cup I_{GH}. \tag{6}$$

Proof. Since $\hat{x}$ is a minimizer for (P), employing Theorem 1, we have

$$0_n \in \{\nabla f(\hat{x})\} + N(S,\hat{x}) = \{\nabla f(\hat{x})\} + \Big(\Gamma(S,\hat{x})\Big)^0. \tag{7}$$

On the other hand, the GGCQ at $\hat{x}$ and (2)-(3) imply that

$$\Big(\Gamma(S,\hat{x})\Big)^0 = \Big(\overline{cone}\Big(\Gamma(S,\hat{x})\Big)\Big)^0 \subseteq \Big(\mathcal{Z}_{I_H}^{I_G}\Big)^{00}.$$

The above inclusion, the bipolar theorem (1), the closedness assumption of $cone\Big(\mathcal{Z}_{I_H}^{I_G}\Big)$, and (7) conclude that

$$0_n \in \{\nabla f(\hat{x})\} + \overline{cone}\Big(\mathcal{Z}_{I_H}^{I_G}\Big) = \{\nabla f(\hat{x})\} + cone\Big(\mathcal{Z}_{I_H}^{I_G}\Big). \tag{8}$$

According to Lemma 1, we obtain from (8) that

$$-\nabla f(\hat{x}) \in \bigcup_{(\mu_r)\in\mathbb{R}^{|I_G|}} \bigcup_{(\eta_s)\in\mathbb{R}^{|I_H|}} \Big(\sum_{r\in I_G} \mu_r \partial_c G_r(\hat{x}) + \sum_{s\in I_H} \eta_s \partial_c H_s(\hat{x})\Big),$$

which concludes that there exist some $\mu_r \in \mathbb{R}$ and $\eta_s \in \mathbb{R}$ as $r \in I_G$ and $s \in I_H$ such that

$$-\nabla f(\hat{x}) \in \Big(\sum_{r\in I_G} \mu_r \partial_c G_r(\hat{x}) + \sum_{s\in I_H} \eta_s \partial_c H_s(\hat{x})\Big).$$

For each

$$r \in I \setminus I_G = I_H \cup I_{GH} \quad \text{and} \quad s \in I \setminus I_H = I_G \cup I_{GH},$$

put $\mu_r := 0$ and $\eta_s := 0$. Thus,

$$-\nabla f(\hat{x}) \in \Big(\sum_{r\in I} \mu_r \partial_c G_r(\hat{x}) + \sum_{s\in I} \eta_s \partial_c H_s(vx)\Big).$$

Changing the indices r and s to i in the above inclusion, and setting $\lambda_i^G := \mu_i$ and $\lambda_i^H := \eta_i$ as $i \in I$, we deduce that

$$\begin{cases} -\nabla f(\hat{x}) \in \sum_{i\in I} \Big(\lambda_i^G \partial_c G_i(\hat{x}) + \lambda_i^H \partial_c H_i(\hat{x})\Big), \\ \lambda_i^G = 0, \quad i \in I_H \cup I_{GH}, \quad \text{and} \quad \lambda_i^H = 0, \quad i \in I_G \cup I_{GH}, \end{cases}$$

as required. □

When the functions G_i and H_i as $i \in I$ are continuously differentiable, (5)-(6) were named the strongly stationary condition for (P) in [9]. Hence we also call them strongly stationary condition.

Now, we state a new optimality condition for (P), and we call it parametric stationary (P-stationary, in short) condition.

Theorem 3. (P-Stationary Condition): Suppose that $\hat{x}$ is an optimal solution of (P) and the PGCQ$_{I_2}^{I_1}$ holds at $\hat{x}$. If $cone\Big(\mathcal{Z}_{I_H \cup I_2}^{I_G \cup I_1}\Big)$ is closed, then there exist some multipliers $\lambda_i^G, \lambda_i^H \in \mathbb{R}$ as $i \in I$, such that

$$-\nabla f(\hat{x}) \in \sum_{i\in I} \Big(\lambda_i^G \partial_c G_i(\hat{x}) + \lambda_i^H \partial_c H_i(\hat{x})\Big), \tag{9}$$

$$\lambda_i^G = 0, \quad i \in I_H \cup I_2, \quad \text{and} \quad \lambda_i^H = 0, \quad i \in I_G \cup I_1. \tag{10}$$

Proof. Repeating the proof of (8), we get

$$0_n \in \{\nabla f(\hat{x})\} + \overline{cone}\Big(\mathcal{Z}_{I_H\cup I_2}^{I_G\cup I_1}\Big) = \{\nabla f(\hat{x})\} + cone\Big(\mathcal{Z}_{I_H\cup I_2}^{I_G\cup I_1}\Big),$$

and according to Lemma 1, we obtain

$$cone\Big(\mathcal{Z}_{I_H\cup I_2}^{I_G\cup I_1}\Big) = \bigcup_{(\mu_r)\in\mathbb{R}^{|I_G\cup I_1|}} \bigcup_{(\eta_s)\in\mathbb{R}^{|I_H\cup I_2|}} \Big(\sum_{r\in I_G\cup I_1} \mu_r \partial_c G_r(\hat{x}) + \sum_{s\in I_H\cup I_2} \eta_s \partial_c H_s(\hat{x})\Big).$$

Using the above two relations and following the proof of Theorem 2 lead us to the required result. □

In what follows, we will show that the parametric stationary condition, presented in Theorem 3, gives rise to two other stationary conditions that are presented in [9] for smooth MPSCs. First, we present the M-stationary condition as follows ("M" is an abbreviation for Mordukhovich):

Theorem 4. (M-Stationary Condition): Suppose that $\hat{x}$ is an optimal solution of (P) and the PGCQ$_{I_2}^{I_1}$ holds at $\hat{x}$. If $cone\Big(\mathcal{Z}_{I_H\cup I_2}^{I_G\cup I_1}\Big)$ is closed, then there exist some multipliers $\lambda_i^G, \lambda_i^H \in \mathbb{R}$ as $i \in I$, such that

$$-\nabla f(\hat{x}) \in \sum_{i\in I}\Big(\lambda_i^G \partial_c G_i(vx) + \lambda_i^H \partial_c H_i(\hat{x})\Big), \tag{11}$$

$$\lambda_i^G = 0, \quad i \in I_H, \quad \text{and} \quad \lambda_i^H = 0, \quad i \in I_G, \tag{12}$$

$$\lambda_i^G \lambda_i^H = 0, \qquad i \in I_{GH}. \tag{13}$$

Proof. Since the inclusion (11) is the same as inclusion (9), and the relation (12) obviously results from relation (10), it suffices to prove only equation (13). Let $i \in I$. Owing to $I = I_1 \cup I_2$, we conclude that $i \in I_1$ or $i \in I_2$, and so, $\lambda_i^H = 0$ or $\lambda_i^G = 0$, respectively. Thus, $\lambda_i^G \lambda_i^H = 0$, and the proof is complete. □

If in Theorem 3 we put $I_1 = I_2 = I_{GH}$, we get the following theorem, named weakly stationary (W-stationary, in short) condition [9] in smooth case.

Theorem 5. (W-Stationary Condition): Suppose that $\hat{x}$ is an optimal solution of (P) and the PGCQ$_{I_{GH}}^{I_{GH}}$ are satisfied at $\hat{x}$. If $cone\Big(\mathcal{Z}_{I_H\cup I_{GH}}^{I_G\cup I_{GH}}\Big)$ is closed, then there exist some multipliers $\lambda_i^G, \lambda_i^H \in \mathbb{R}$ as $i \in I$, such that

$$-\nabla f(\hat{x}) \in \sum_{i\in I}\Big(\lambda_i^G \partial_c G_i(\hat{x}) + \lambda_i^H \partial_c H_i(\hat{x})\Big),$$

$$\lambda_i^G = 0, \quad i \in I_H \cup I_{GH}, \quad \text{and} \quad \lambda_i^H = 0, \quad i \in I_G \cup I_{GH}.$$

The following implications are straightforward consequences of the aforementioned definitions of stationary conditions:

S-Stationarity ⇒ P-Stationarity ⇒ M-Stationarity ⇒ W-Stationarity.

The following example thoroughly and accurately analyzes the strict relationships between the various constraint qualifications and different stationary conditions expressed in the present article.

Example 1. Put in problem (P),

$$n := 2, \quad x := (x_1, x_2) \in \mathbb{R}^2, \quad f(x) := 2x_1 + 3x_2,$$

$$G_1(x) := x_1, \qquad H_1(x) := x_2,$$

$$G_2(x) := 1, \quad H_2(x) := (x_1^2 - x_2) + |x_1^2 - x_2|.$$

It is easy to see that $S = \{0\} \times [0, +\infty)$ and that $\hat{x} := 0_2$ is an optimal solution to the problem. Also, a short calculation shows that:

$$I_G = \emptyset, \quad I_H = \{2\}, \qquad I_{GH} = \{1\},$$

$$\nabla f(0_2) = (2,3), \quad \partial_c G_1(0_2) = \{(1,0)\}, \quad \partial_c H_1(0_2) = \{(0,1)\},$$

$$\partial_c G_2(0_2) = \{0_2\}, \qquad \partial_c H_2(0_2) = \{0\} \times \Big(-1 + [-1,1] \Big) = \{0\} \times [-2,0].$$

$$\Big(\mathcal{G}^{I_G \cup I_1} \Big)^{\perp} = \{(1,0)\}^{\perp} = \{0\} \times \mathbb{R},$$

$$\Big(\mathcal{H}^{I_H \cup I_2} \Big)^{\perp} = \Big(\{0\} \times [-2,0] \Big)^{\perp} = \mathbb{R} \times \{0\},$$

$$cone\Big(\mathcal{Z}^{I_G \cup I_1}_{I_H \cup I_2} \Big) = \mathbb{R}^2,$$

$$\Big(\mathcal{G}^{I_G \cup I_1} \Big)^{\perp} \cap \Big(\mathcal{H}^{I_H \cup I_2} \Big)^{\perp} = \{0_2\} \subseteq S = \overline{cone}\Big(\Gamma(S, 0_2) \Big).$$

Thus, $\text{PGCQ}^{I_1}_{I_2}$ holds at $\hat{x}$ and $cone\Big(\mathcal{Z}^{I_G \cup I_1}_{I_H \cup I_2} \Big)$ is closed, i.e., all hypotheses of Theorem 3 hold. Obviously, we can find some scalars λ_i^G and λ_i^H as $i = 1,2$ satisfying (9)-(10). In fact, taking

$$\lambda_1^G := -2, \quad \lambda_1^H := 0, \quad \lambda_2^G := 0, \quad \lambda_2^H := \frac{3}{2},$$

we have

$$(-2,-3) \in \lambda_1^G \{(1,0)\} + \lambda_1^H \{(0,1)\} + \lambda_2^G \{0_2\} + \lambda_2^H \Big(\{0\} \times [-2,0] \Big).$$

This means the P-stationary condition, M-stationary condition, and W-stationary condition are satisfied at $\hat{x}$. It should be observed that since

$$\Big(\mathcal{G}^{I_G} \Big)^{\perp} \cap \Big(\mathcal{H}^{I_H} \Big)^{\perp} = \mathbb{R}^2 \cap \Big(\mathbb{R} \times \{0\} \Big) = \mathbb{R} \times \{0\} \not\subseteq S = \overline{cone}\Big(\Gamma(S, 0_2) \Big),$$

the GGCQ does not hold at $\hat{x}$, and the S-stationary condition (5)-(6) is not valid. In fact, there are not λ_i^G and λ_i^H as $i = 1,2$ satisfying

$$(-2,-3) \in \lambda_1^G \{(1,0)\} + \lambda_1^H \{(0,1)\} + \lambda_2^G \{0_2\} + \lambda_2^H \Big(\{0\} \times [-2,0] \Big),$$

$$\lambda_1^G = \lambda_2^G = \lambda_1^H = 0.$$

As the final point of this article, we introduce a broad and important class of MPSCs that satisfy parametric Guignard constraint qualification at all of their feasible points but do not necessarily satisfy generalized Guignard constraint qualification. It is noteworthy that in this category of MPSCs, none of the appearing functions are necessarily convex.

Consider the following optimization problem with fractional quadratic constraints on $\mathbb{R}$:

$$(Q) \qquad \min \quad f(x)$$
$$s.t. \quad \frac{(a_i x + b_i)(a_i' x + b_i')}{(c_i x + d_i)(c_i' x + d_i')} = 0, \qquad i \in I,$$
$$x \in \mathbb{R},$$

where the numbers $a_i, a_i', c_i, c_i', b_i, b_i', d_i, d_i' \in \mathbb{R}$ are fixed such that $(a_i, b_i) \neq 0_2 \neq (c_i, d_i)$ as $i \in I := \{1, \ldots, m\}$, and where $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. The feasible set of (Q) is denoted by $\hat{S}$. This problem is a MPSC with the following data:

$$G_i(x) = \frac{a_i x + b_i}{c_i x + d_i}, \quad \text{and} \quad H_i(x) = \frac{a_i' x + b_i'}{c_i' x + d_i'}.$$

Since G_i and H_i, for $i \in I$, are continuously differentiable on their open domain, they are locally Lipschitz near each point on their domains. Let $\hat{x} \in \mathbb{R}^n$ be a feasible point for (Q), and $J_1, J_2 \subseteq I_{GH}$ be given with $J_1 \cup J_2 = I_{GH}$. The definition of polar cones implies that $0 \in A^0$ for all $A \subseteq \mathbb{R}$. Thus, $\left(\mathcal{Z}^{I_G \cup J}_{I_H \cup I_{GH} \setminus J}\right)^0 \neq \emptyset$. Let $w \in \left(\mathcal{Z}^{I_G \cup J}_{I_H \cup I_{GH} \setminus J}\right)^0$ be chosen arbitrarily. According to

$$w \in \left(\mathcal{G}^{I_G \cup J_1}\right)^{\perp} \cap \left(\mathcal{H}^{I_H \cup J_2}\right)^{\perp},$$

and

$$\nabla G_i(\hat{x}) = \frac{a_i d_i - b_i c_i}{(c_i \hat{x} + d_i)^2}, \quad \text{and} \quad \nabla H_i(\hat{x}) = \frac{a_i' d_i' - b_i' c_i'}{(c_i' \hat{x} + d_i')^2},$$

we have

$$\begin{cases} \left(\dfrac{a_i d_i - b_i c_i}{(c_i \hat{x} + d_i)^2}\right) w = 0, & \text{for } i \in I_G \cup J_1, \\ \left(\dfrac{a_i' d_i' - b_i' c_i'}{(c_i' \hat{x} + d_i')^2}\right) w = 0, & \text{for } i \in I_H \cup J_2. \end{cases} \tag{14}$$

The first equality in (14) concludes, for each $t \geq 0$ and for $i \in I_G \cup J_1$, that

$$(a_i d_i - b_i c_i)(\hat{x} + tw - \hat{x}) = 0 \implies a_i d_i (\hat{x} + tw) + b_i c_i \hat{x} = b_i c_i (\hat{x} + tw) + a_i d_i \hat{x}.$$

By adding the two sides of latter equality with $a_i c_i \hat{x}(\hat{x} + tw) + b_i d_i$ and factoring in the appropriate expressions, we obtain that

$$a_i(\hat{x} + tw)(c_i \hat{x} + d_i) + b_i(c_i \hat{x} + d_i) = c_i(\hat{x} + tw)(a_i \hat{x} + b_i) + d_i(a_i \hat{x} + b_i),$$

and so

$$\Big(a_i(\hat{x}+tw)+b_i\Big)(c_i\hat{x}+d_i)=\Big(c_i(\hat{x}+tw)+d_i\Big)(a_i\hat{x}+b_i).$$

This means that

$$\frac{a_i(\hat{x}+tw)+b_i}{c_i(\hat{x}+tw)+d_i}=\frac{a_i\hat{x}+b_i}{c_i\hat{x}+d_i}=0, \tag{15}$$

where the last equality holds for $i\in I_G\cup J_1$. Similarly, from the second equality of (14), for each $i\in I_H\cup J_2$ we deduce that

$$\frac{a'_i(\hat{x}+tw)+b'_i}{c'_i(\hat{x}+tw)+d'_i}=\frac{a'_i\hat{x}+b'_i}{c'_i\hat{x}+d'_i}=0. \tag{16}$$

Owing to (15)-(16) and the fact that $I=I_G\cup I_H\cup J_1\cup J_2$, we have

$$\frac{a_i(\hat{x}+tw)+b_i}{c_i(\hat{x}+tw)+d_i}\cdot\frac{a'_i(\hat{x}+tw)+b'_i}{c'_i(\hat{x}+tw)+d'_i}=0,\quad \text{for all } i\in I.$$

Thus, $\hat{x}+tw\in\hat{S}$ for all $t\geq 0$, and so $w\in\Gamma(\hat{S},\hat{x})$. Summarizing, we proved that

$$\left(\mathcal{Z}_{I_H\cup J_2}^{I_G\cup J_1}\right)^0\subseteq\Gamma(\hat{S},\hat{x})\subseteq\overline{cone}\Big(\Gamma(\hat{S},\hat{x})\Big).$$

Therefore, $\text{PGCQ}_{J_2}^{J_1}$ holds at every point in $\hat{S}$. On the other hand, since

$$\mathcal{Z}_{I_H\cup J_2}^{I_G\cup J_1}=\left\{\pm\frac{a_id_i-b_ic_i}{(c_i\widetilde{x}+d_i)^2}\mid i\in I_G\cup J_1\right\}\cup\left\{\pm\frac{a'_id'_i-b'_ic'_i}{(c'_i\widetilde{x}+d'_i)^2}\mid i\in I_H\cup J_2\right\},$$

then, the number of members of $\mathcal{Z}_{I_H\cup J_2}^{I_G\cup J_1}$ is not more than $2m$. So, according to the well-known result that says "the convex cone of each finite set in $\mathbb{R}^n$ is closed" [4], we conclude that $cone\left(\mathcal{Z}_{I_H\cup J_2}^{I_G\cup J_1}\right)$ is closed. Thus, P-stationary condition is satisfied at each optimal solution of (Q) by Theorems 3 and 4, respectively. It should be noted that the GGCQ does not necessarily hold at all feasible points of (Q) and S-stationary condition is not necessarily satisfied at all optimal solutions of (Q).

Remark 3. Here we present another proof to reach (15)-(16) out of (14).

If $w=0$, then (15)-(16) hold clearly. Thus, we suppose that $w\neq 0$ satisfies (14). Hence,

$$\begin{cases}\dfrac{a_id_i-b_ic_i}{(c_i\hat{x}+d_i)^2}=0, & i\in I_G\cup J_1,\\[2ex] \dfrac{a'_id'_i-b'_ic'_i}{(c'_i\hat{x}+d'_i)^2}=0, & i\in I_H\cup J_2,\end{cases}\Longrightarrow\begin{cases}a_id_i=b_ic_i, & i\in I_G\cup J_1,\\ a'_id'_i=b'_ic'_i, & i\in I_H\cup J_2.\end{cases}$$

This means that there exist $k,k'\in\mathbb{R}$ such that

$$\begin{cases}\dfrac{a_ix+b_i}{c_ix+d_i}=k, & i\in I_G\cup J_1,\ x\in\hat{S},\\[2ex] \dfrac{a'_ix+b'_i}{c'_ix+d'_i}=k', & i\in I_H\cup J_2,\ x\in\hat{S}.\end{cases}$$

This means that $G_i(x)$ as $i \in I_G \cup J_1$ and $H_i(x)$ as $i \in I_H \cup J_2$ are independent of x, and since

$$\begin{cases} \dfrac{a_i\hat{x}+b_i}{c_i\hat{x}+d_i} = 0, & i \in I_G \cup J_1, \\ \\ \dfrac{a'_i\hat{x}+b'_i}{c'_i\hat{x}+d'_i} = 0, & i \in I_H \cup J_2, \end{cases}$$

we have $k = k' = 0$, and so

$$\begin{cases} \dfrac{a_i(\hat{x}+tw)+b_i}{c_i(\hat{x}+tw)+d_i} = 0, & i \in I_G \cup J_1, \\ \\ \dfrac{a'_i(\hat{x}+tw)+b'_i}{c'_i(\hat{x}+tw)+d'_i} = 0, & i \in I_H \cup J_2. \end{cases}$$

4 Conclusion

Since MPSC, like “mathematical programming with equilibrium constraints” (MPEC) and “mathematical programming with vanishing constraints” (MPVC), is in the class of optimization problems that include multiplicative constraints, in order to present similar works [1, 2], and [10, 11] which consider nonsmooth MPECs and MPVCs, respectively, this article deals with the nonsmoothness of the results of [9].

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