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Research Article

Optimal Control Problems: Convergence and Error Analysis in Reproducing Kernel Hilbert Spaces

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Abstract. In this article, we offer an efficient method to find an approximate solution for quadratic optimal control problems. The approximate solution is offered in a finite series form in reproducing kernel space. The convergence of proposed method is analyzed under some hypotheses which provide the theoretical basis of the proposed method for solving quadratic optimal control problems. Furthermore, in this study, we investigate the application of the proposed method to obtain the solution of equations that have formally been solved using Pontryagin's maximum principle. Moreover, many different types of quadratic optimal control problems are considered prototype examples. The obtained results demonstrate that the proposed method is truly effective and convenient to obtain the analytic and approximate solutions of quadratic optimal control problems.

Keywords. Optimal control problem, Pontryagin's maximum principle, Convergence, Reproducing kernel Hilbert space.

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1 Introduction

Optimal control is a particular branch of modern control theory which has been broadly applied in various fields including aviation systems [12], robotic [40], biomedicine [21], etc. Recently, different types of computational techniques have been expounded for solving optimal control problems (OCPs). For instance, we can mention the following papers. Salim [35] presented a method based on the parameterization of both state and control variables, the authors of [5] applied the relaxed descent method to approximate solutions for OCPs, Nemati et al. [29] applied the Bernstein polynomials with the fractional operational matrix to obtain the solution of a class of fractional (OCPs), Gholami Baladezaei [20] applied a $1/G'$-expansion technique for solving nonlinear (OCPs), Nezhadhosein [30] used the Haar matrix equations to obtain the solution of continuous time-variant linear-quadratic OCPs, Chryssoverghi et al. [6] applied discretization methods to solve the OCPs with some state constraints, in [22] the authors applied the modal series method to study infinite horizon nonlinear control problems, EL-Gindy et al. [10] presented an alternative technique for solving controlled Duffing oscillator problems, the authors of [34] the applied differential transform method to approximate solutions for the linear OCPs. In [24], Kafash et al. applied Chebyshev polynomials to obtain some suitable algorithms for solving the OCPs, authors of [38] applied a Chebyshev technique for solving nonlinear OCPs, Betts et al. [2] used techniques to show that a nonconvergent Runge–Kutta method converges for OCPs, the authors of [25] applied the pseudospectral method to analyze the solution of OCPs, Canuto et al. [4] applied a pseudospectral method for solving infinitely smooth and well-behaved problems, the examples of Radau pseudospectral method to numerically solve OCPs are given in [11]. Also, the authors of [41] used the basic variational iteration method to obtain approximate solution of linear quadratic OCPs.

The concept of reproducing kernel was first applied by Zaremba [42] to obtain the approximate solution of boundary value problems for harmonic functions (see [1] for more details). Researchers have been investigating this theory for constructing approximate solutions to fractal interpolation [39, 3], second-order three-point boundary value problems with the property of singularity [13], singularly perturbed boundary value problems [14], nonlocal fractional boundary value problems [17], Riccati differential equations [18], nonlinear delay differential equations of fractional order [19], BlackScholes equation [37], and nonlinear differential-difference equations. The authors of [8, 27, 7] examined the reproducing kernel to derive solution of some partial differential equations. The book [9] provides a wide range of reproducing kernel methods which have been used to solve various model problems.

The main idea of this study is to find an approximate solution of a quadratic OCPs in reproducing kernel spaces. Some ordinary capabilities of the method lie in the following formations. The current method is mesh-free, easily executed and useful for various boundary conditions and the obtained approximate solution converges uniformly to the exact solution.

This paper is organized as follows. The statement of quadratic OCPs is described in Section 2. In Section 3, we express some nearly new definitions used in this paper. In Section 4, we investigate and analyze the derived results to the proposed method.

In Section 5, four numerical examples are presented to illustrate the accuracy and efficiency of our method. Eventually, we provide some concluding remarks in Section 6.

2 Statement of the Problem

Consider the following single-input n-state dynamic system

$$\begin{cases} \dot{x}(t) = f(t, x(t)) + Bv(t), \quad 0 \leq t \leq \mathrm{T}, \\ x(0) = x_0, \end{cases} \tag{1}$$

where $x(t) = (x_1(t), ..., x_n(t))^T \in \Re^n$ is the state vector, $v(t) \in \Re$ is the control function and $x_0 \in \Re^n$ is the initial state vector at $t = 0$. Moreover, $f(t, x(t)) \in \Re^n$ is a continuously differentiable function in all arguments and $B \in \Re^{n\times 1}$ is a real constant vector.
Now the unconstrained OCP can be stated as finding the optimal control law $v(t)$, minimizing the quadratic objective functional

$$U[x(t), v(t)] = \frac{1}{2} x^T(\mathrm{t}) K x(\mathrm{t}) + \frac{1}{2}\int_0^{\mathrm{t}} (x^T(s) S x(s) + v^T(s) R v(s)) ds. \tag{2}$$

Now the unconstrained optimal control problem can be stated as finding the optimal control law $v(.)$ that minimizes the quadratic objective functional (2) subject to the control system (1), where K and S are symmetric positive semi-definite $n \times n$ matrices and R is a positive constant.

Consider Hamiltonian control system (1) as

$$\mathrm{H}[t, x(t), v(t), p(t)] = \frac{1}{2}[x^T(t) S x(t) + v^T(t) R v(t)] + p^T(t)[f(t, x(t)) + Bv(t)],$$

where $p(t) \in \Re^n$ is the co-state vector with the i-th component $p_i(t), i = 1, 2, ..., n$.
Using Pontryagin's maximum principle [31], the optimality condition for system (1) can be described by the subsequent equations

$$\begin{cases} \dot{x}(t) = f(t, x(t)) - (BR^{-1}B^T)p(t), \\ \dot{p}(t) = -Sx(t) - g(t, x(t), p(t)), \quad g(t, x(t), p(t)) = [\frac{\partial f(t, x(t))}{\partial x}]^T p(t), \\ x(0) = x_0, \quad 0 \leq t \leq \mathrm{T}, \end{cases} \tag{3}$$

with the terminal condition $p(\mathrm{T}) = Kx(\mathrm{T})$. From this the optimal control law is determined by $v(t) = -R^{-1}B^T p(t)$.
Using change of variables $y(t) = x(t) - x_0$ and $z(t) = p(t) - p(\mathrm{T})$, the aforementioned system can further be converted into the following form

$$\begin{cases} \dot{y}(t) + (BR^{-1}B^T)z(t) = F(t, y(t)) - (BR^{-1}B^T)p(\mathrm{T}), \\ \dot{z}(t) + Sy(t) = -Sx_0 - G(t, y(t), z(t)), \\ y(0) = 0, \; z(\mathrm{T}) = 0, \quad 0 \leq t \leq \mathrm{T}, \end{cases} \tag{4}$$

where

$$\begin{cases} y(t) = (y_1(t),...,y_n(t))^T, \\ z(t) = (z_1(t),...,z_n(t))^T, \\ F(t,y(t)) = (F_1(t,y(t)),...,F_n(t,y(t)))^T = f(t,y(t)+x_0), \\ G(t,y(t),z(t)) = (G_1(t,y(t),z(t)),...,G_n(t,y(t),z(t)))^T \\ \qquad\qquad = g(t,y(t)+x_0,z(t)+p(\mathrm{T})). \end{cases} \tag{5}$$

The functions $F(t,y(t))$ and $G(t,y(t),z(t))$ can be divided into two parts, a linear part and a nonlinear part and therefore (4) can be rewritten in the following form

$$\begin{cases} \dot{y}(t) + (BR^{-1}B^T)z(t) = F_0(t) + F_L(t,y(t)) + F_N(t,y(t)), \\ \dot{z}(t) + Sy(t) = G_0(t) + G_L(t,y(t),z(t)) + G_N(t,y(t),z(t)), \\ y(0) = 0, \ z(\mathrm{T}) = 0, \quad 0 \le t \le \mathrm{T}, \end{cases} \tag{6}$$

where F_N and G_N are two nonlinear functions and F_L and G_L are two linear functions and

$$\begin{cases} F(t,y(t)) - (BR^{-1}B^T)p(\mathrm{T}) = F_0(t) + F_L(t,y(t)) + F_N(t,y(t)), \\ -Sx_0 - G(t,y(t),z(t)) = G_0(t) + G_L(t,y(t),z(t)) + G_N(t,y(t),z(t)). \end{cases} \tag{7}$$

3 Hilbert Function Spaces

We give some basic definitions and properties of Hilbert function spaces, and then we construct some Hilbert function spaces which are used in the proceeding sections.

Definition 1. (see [15]). A Hilbert space H of functions $y : E \to \Re$ is called a Hilbert function space if for each $t \in E$, there exists a positive constant c_t such that $|y(t)| \le c_t \|y\|_H$ for all y in H.

Definition 2. (see [15]). The inner product spaces $W_2^{2,i}[0,\mathrm{T}], i = 0,1,$ of real-valued functions are defined as

$$\begin{aligned} W_2^{2,1}[0,\mathrm{T}] \quad = \quad & \{y(t)| \text{ where } y \text{ and } \dot{y} \text{ are absolutely continuous functions,} \\ & \ddot{y} \in L^2[0,\mathrm{T}], \ y(0) = 0, \ t \in [0,\mathrm{T}]\}, \end{aligned} \tag{8}$$

and

$$\begin{aligned} W_2^{2,2}[0,\mathrm{T}] \quad = \quad & \{y(t)| \text{ where } y \text{ and } \dot{y} \text{ are absolutely continuous functions,} \\ & \ddot{y} \in L^2[0,\mathrm{T}], \ y(\mathrm{T}) = 0, \ t \in [0,\mathrm{T}]\}. \end{aligned} \tag{9}$$

Also, the specific inner product in $W_2^{2,i}[0,\mathrm{T}]$ $i = 0,1$ is of the subsequent form

$$\langle y(t), \widetilde{y}(t) \rangle = y(0)\widetilde{y}(0) + \int_0^{\mathrm{t}} \ddot{y}(s)\ddot{\widetilde{y}}(s)ds, \tag{10}$$

and the norm in the inner product space $\|y(t)\|_{W_2^{2,i}}$ is given by

$$\|y(t)\|_{W_2^{2,i}} = \sqrt{\langle y(t), y(t) \rangle_{W_2^{2,i}}}, \tag{11}$$

where $y, \widetilde{y} \in W_2^{2,i}[0,\mathrm{T}]$.

It is clear that $W_2^{2,i}[0,\mathrm{T}]$ is a Hilbert space.

Definition 3. (see [15]). The inner product space $W_2^1[0,\mathrm{T}]$ is defined as

$$\begin{aligned} W_2^1[0,\mathrm{T}] &= \{y(t)| \text{ where } y \text{ is an absolutely continuous real-valued function}, \\ & \quad y,\dot{y} \in L^2[0,\mathrm{T}],\ t \in [0,\mathrm{T}]\}. \end{aligned} \tag{12}$$

The inner product in $W_2^1[0,\mathrm{T}]$ is of the form

$$\langle y(t),\widetilde{y}(t)\rangle = \int_0^{\mathrm{t}} (y(s)\widetilde{y}(s) + \dot{y}(s)\dot{\widetilde{y}}(s))ds, \tag{13}$$

and the norm $\|y(t)|_{W_2^1}$ is defined by

$$\|y(t)\|_{W_2^1} = \sqrt{\langle y(t),y(t)\rangle_{W_2^1}}, \tag{14}$$

where $y(t),\widetilde{y}(t) \in W_2^1[0,\mathrm{T}]$.

Definition 4. (see [9]). Let H be a real Hilbert space of functions defined on a set E. The specific product is denoted by $\langle x,x\rangle_H$. Let $\|x\| = \sqrt{\langle x,x\rangle_H}$ be the norm in the Hilbert space H, for $x,y \in H$. The function $R : E\times E \to \Re$ is called a reproducing kernel of H if the following conditions are satisfied:

(i) t, $R_t(s) = R(t,s)$ as a real-valued function of s belongs to H.

(ii) For every $x \in H$ and $t \in E$, we have $\langle x(.),R_t(.)\rangle_H$. This is called a reproducing property.

Theorem 1. (see [15]). The space $W_2^{2,0}[0,\mathrm{T}]$ consisting of real-valued functions is a Hilbert function space and the reproducing kernel for this space can be explained as follows

$$R_s^0(t) = \begin{cases} C_s^0(t) = \sum_{i=1}^4 c_i(s)t^{i-1}, & t \le s, \\ D_s^0(t) = \sum_{i=1}^4 d_i(s)t^{i-1}, & t > s, \end{cases} \tag{15}$$

where, the coefficients $c_i(s),\ d_i(s),\ i = 1,\dots,4,$ are characterized as follows

- $R_s^0(0) = 0,\ \frac{\partial R_s^0(0)}{\partial t} - \frac{\partial^2 R_s^0(0)}{\partial t^2} = 0,\ (\frac{\partial^3 C_s^0(t)}{\partial t^3}|_{t=s} - \frac{\partial^3 D_s^0(t)}{\partial t^3}|_{t=s}) = 1,$
- $\frac{\partial^{3-i} R_s^0(\mathrm{T})}{\partial t^{3-i}} = 0,\ i = 0,1,$
- $\frac{\partial^i C_s^0(t)}{\partial t^i}|_{t=s} = \frac{\partial^i D_s^0(t)}{\partial t^i}|_{t=s},\quad i = 0,1,2.$

Then, by using the features of the reproducing kernel $R_t^0(s)$, the solution of above drivative equations is computed.

Theorem 2. (see [39]). The space $W_2^{2,1}[0,\mathrm{T}]$ consisting of real-valued functions is a Hilbert function space and the corresponding kernel for this space can be explained as follows

$$R_s^1(t) = \begin{cases} \widetilde{C}_s^1(t) = \sum_{i=1}^4 \widetilde{c}_i(s)t^{i-1}, & t \le s, \\ \widetilde{D}_s^1(t) = \sum_{i=1}^4 \widetilde{d}_i(s)t^{i-1}, & t > s, \end{cases} \tag{16}$$

where, the coefficients $\widetilde{c}_i(s),\ \widetilde{d}_i(s),\ i = 1,\dots,4,$ are characterized as follows

- $R_s^1(\mathrm{T}) = 0,\ \frac{\partial^3 R_s^1(\mathrm{T})}{\partial t^3} = 0,\ (\frac{\partial^3 \widetilde{C}_s^1(t)}{\partial t^3}|_{t=s} - \frac{\partial^3 \widetilde{D}_s^1(t)}{\partial t^3}|_{t=s}) = 1,$
- $\frac{\partial^i R_s^1(0)}{\partial t^i} - (-1)^{1-i}\frac{\partial^{3-i} R_s^1(0)}{\partial t^{3-i}} = 0,\ i = 0,1,$
- $\frac{\partial^i \widetilde{C}_s^1(t)}{\partial t^i}|_{t=s} = \frac{\partial^i \widetilde{D}_s^1(t)}{\partial t^i}|_{t=s},\quad i = 0,1,2.$

Then, by using the features of the reproducing kernel $R_t^1(s)$, the solution of above drivative equations is computed.

3.1 The inner product space $\widetilde{W}[0,\mathrm{T}]$

The inner product space $\widetilde{W}[0,\mathrm{T}]$ is defined as

$$\widetilde{W}[0,\mathrm{T}] = \underbrace{W_2^{2,0}[0,\mathrm{T}]\bigoplus\cdots\bigoplus W_2^{2,0}[0,\mathrm{T}]}_{n}\bigoplus\underbrace{W_2^{2,1}[0,\mathrm{T}]\bigoplus\cdots\bigoplus W_2^{2,1}[0,\mathrm{T}]}_{n},$$

$$\widetilde{W}[0,\mathrm{T}] = \{(y_1,...,y_n,z_1,...,z_n)^T | y_i \in W_2^{2,0}[0,\mathrm{T}], z_i \in W_2^{2,1}[0,\mathrm{T}], i = 1,...,n\}.$$

The specific inner product in $\widetilde{W}[0,\mathrm{T}]$ is of the subsequent form

$$\begin{aligned}&\langle (y_1,...,y_n,z_1,...,z_n)^T,(\widetilde{y}_1,...,\widetilde{y}_n,\widetilde{z}_1,...,\widetilde{z}_n)^T\rangle_{\widetilde{W}}\\&= \sum_{i=1}^{n} < y_i,\widetilde{y}_i >_{W_2^{2,0}} + \sum_{i=1}^{n} < z_i,\widetilde{z}_i >_{W_2^{2,1}},\end{aligned} \tag{17}$$

and the norm $\|(y_1,...,y_n,z_1,...,z_n)^T\|_{\widetilde{W}}$ is denoted by

$$\begin{aligned}&\|(y_1,...,y_n,z_1,...,z_n)^T\|_{\widetilde{W}}\\&= \sqrt{\sum_{i=1}^{n}\|y_i\|^2_{W_2^{2,0}} + \sum_{i=1}^{n}\|z_i\|^2_{W_2^{2,1}}}.\end{aligned} \tag{18}$$

It is easy to verify that $\widetilde{W}[0,\mathrm{T}]$ is a Hilbert space.
Also, $\overline{W}[0,\mathrm{T}] = \bigoplus_{i=1}^{2n} W_2^1$ is a Hilbert space in a similar procedure.

4 Analytical Solution of the System

In this section, we will characterize the analytical solution of system (6) in the space $\widetilde{W}[0,\mathrm{T}]$.

First, consider the following assumptions:
(I) Suppose that the problem (6) has a unique solution.
(II) Let $\Lambda\left(l_{ij} \right)_{2n\times 2n} : \widetilde{W}[0,\mathrm{T}] \longrightarrow \overline{W}[0,\mathrm{T}]$ be a linear operator, where

$$\Lambda(y(t),z(t)) = (\dot{y}(t) + (BR^{-1}B)z(t) - F_L(t,y(t)), \dot{z}(t) + Sy(t) - G_L(t,y(t),z(t))). \tag{19}$$

Then (6) can be converted into the following form

$$\Lambda(y(t),z(t)) = (F_0(t)+F_N(t,y(t)),G_0(t)+G_N(t,y(t),z(t))). \tag{20}$$

Theorem 3. The operator $(l_{ij})_{2n\times 2n}$ is a linear and bounded operator.
Since $(l_{ij})_{2n\times 2n}$ is a linear and bounded operator, then the adjoint operator of $(l_{ij})_{2n\times 2n}$ defined as subsequent form $(l_{ij})_{2n\times 2n}{}^* : \overline{W}[0,T] \longrightarrow \widetilde{W}[0,T]$ is uniquely determined.
Let a countable set $\{t_i\}_{i=1}^{\infty}$ be dense in the intervel $[0,T]$ and $R_s(t)$ be the reproducing kernel of $W_2^1[0,T]$. Now, we set $\rho_{ij}(t) = (l_{ij})_{2n\times 2n}{}^*\Phi_{ij}(t)$, $i = 1,2,...,$ $j = 1,2,...,2n$, where

$$\Phi_{ij}(t) = R_{t_i}(t)\overrightarrow{e}_j = \begin{cases} (R_t(t_i),\underbrace{0,0,...,0}_{2n-1})^T, & j=1,\\ (0,R_t(t_i),\underbrace{0,...,0}_{2n-2})^T, & j=2,\\ \vdots & \\ (\underbrace{0,0,...,0}_{2n-1},R_t(t_i))^T, & j=2n. \end{cases} \tag{21}$$

Lemma 1. (see [16]). For each $i \in \{1,2,3,...\}$ and each $j \in \{1,2,...,2n\}$, $\rho_{ij} \in \widetilde{W}[0,T]$.

Lemma 2. (see [23]) If the set $\{t_i\}_{i=1}^{\infty}$ is dense in the intervel $[0,T]$, then the system $\{\rho_{ij}(t)\}_{(1,1)}^{(\infty,2n)}$ is independent in $\widetilde{W}[0,T]$.

Theorem 4. (see [23]) Suppose that the set $\{t_i\}_{i=1}^{\infty}$ is dense in the intervel $[0,T]$ and the solution of system (20) is unique, then $\{\rho_{ij}(t)\}_{(1,1)}^{(\infty,2n)}$ is the complete system of $\widetilde{W}[0,T]$ and

$$\rho_{ij}(t) = ((l_{j1})_s R_s^0(t)|_{s=t_i},...,(l_{jn})_s R_s^0(t)|_{s=t_i},(l_{jn+1})_s R_s^1(t)|_{s=t_i},...,(l_{j2n})_s R_s^1(t)|_{s=t_i})^T,$$

and the subscript s of the operator $l_{jj}, j = 1,2,...,2n,$ indicates that we can apply this operator to the function of s.

4.1 The Linear problem

If (20) is linear, that is $F_N(t,y(t)) = G_N(t,y(t),z(t)) = (\underbrace{0,0,...0}_{n})^T$, then (20) can be rewritten as follows.

$$\Lambda(y(t),z(t)) = (F_0(t),G_0(t)). \tag{22}$$

The exact solution and approximate solution can be derived by using the following theorem.

Theorem 5. Suppose that $(y(t),z(t)) \in \widetilde{W}[0,T]$, then

$$(y(t),z(t))=\sum_{i=1}^{\infty}\sum_{j=1}^{2n}C_{ij}\rho_{ij}(t), \tag{23}$$

where

$$\sum_{i=1}^{m}\sum_{j=1}^{2n}C_{ij}\Lambda\rho_{ij}(t_k)=(F_0(t_k),G_0(t_k)),\ k=1,2,...,m, \tag{24}$$

Proof. The system $\{\rho_{ij}(t)\}_{(1,1)}^{(\infty,2n)}$ is complete in $\widetilde{W}[0,\mathrm{T}]$, then

$$(y(t),z(t))=\sum_{i=1}^{\infty}\sum_{j=1}^{2n}C_{ij}\rho_{ij}(t). \tag{25}$$

Now, by the m-term intercept of (25), the approximate solution is presented by

$$Q_m(y(t),z(t))=(y^m(t),z^m(t))=\sum_{i=1}^{m}\sum_{j=1}^{2n}C_{ij}\rho_{ij}(t), \tag{26}$$

where $Q_m:\ \widetilde{W}[0,\mathrm{T}]\to\{\rho_{ij}(t)\}_{(1,1)}^{(m,2n)}$ is an orthogonal projection operator.
It follows that

$$\begin{aligned}\Lambda(y^m(t_k),z^m(t_k)) &= \sum_{j=1}^{2n}\langle\Lambda(y^m(t),z^m(t)),\Phi_{kj}(t)\rangle_{\widetilde{W}}\vec{e}_j\\ &=\sum_{j=1}^{2n}\langle(y^m(t),z^m(t)),\Lambda^*\Phi_{kj}(t)\rangle_{\widetilde{W}}\vec{e}_j=\sum_{j=1}^{2n}\langle Q_m(y(t),z(t)),\rho_{kj}(t)\rangle_{\widetilde{W}}\vec{e}_j\\ &=\sum_{j=1}^{2n}\langle(y(t),z(t)),Q_m\rho_{kj}(t)\rangle_{\widetilde{W}}\vec{e}_j=\sum_{j=1}^{2n}\langle(y(t),z(t)),\rho_{kj}(t)\rangle_{\widetilde{W}}\vec{e}_j\\ &=\sum_{j=1}^{2n}\langle\Lambda(y(t),z(t)),\Phi_{kj}(t)\rangle_{\widetilde{W}}\vec{e}_j=\Lambda(y(t_k),z(t_k)),k=1,2,...,m.\end{aligned} \tag{27}$$

Therefore,

$$\sum_{i=1}^{m}\sum_{j=1}^{2n}C_{ij}\Lambda\rho_{ij}(t_k)=(F_0(t_k),G_0(t_k)),\ k=1,2,...,m. \tag{28}$$

Then, the approximate solution $(y(t),z(t))$ can be obtained by

$$(y^m(t),z^m(t))=Q_m(y(t),z(t))=\sum_{i=1}^{m}\sum_{j=1}^{2n}C_{ij}\rho_{ij}(t), \tag{29}$$

where the coefficients $C_{ij},\ i=1,2,...,n,\ j=1,2,...,2n$, can be determined by (28). □

4.1.1 Convergence analysis

Theorem 6. Let $\{t_i\}_{i=1}^{\infty}$ be dense in the intervel $[0,\mathrm{T}]$ and $(y(t),z(t))$ be the solution of (22), then the approximate solution $(y^m(t),z^m(t))$ and its derivative $(\dot{y}^m(t),\dot{z}^m(t))$ are uniformly convergent to the exact solution $(y(t),z(t))$ and $(\dot{y}(t),\dot{z}(t))$, respectively.

Proof. Noting that $\widetilde{W}[0,\mathrm{T}]$ is a Hilbert space, we obtain

$$\|y_i(t)-y_i^m(t)\|_{W_2^{2,0}} \to 0, \quad as \quad m\to\infty, \quad i=1,\dots,n, \tag{30}$$

$$\|z_i(t)-z_i^m(t)\|_{W_2^{2,1}} \to 0, \quad as \quad m\to\infty, \quad i=1,\dots,n. \tag{31}$$

On the other hand, it holds that

$$\begin{aligned} |y_i(s)-y_i^m(s)| &= |<y_i(.)-y_i^m(.),R_s^0(.)>_{W_2^{2,0}}| \\ &\le \nu_i^1\|y_i-y_i^m\|_{W_2^{2,0}}, \quad i=1,\dots,n, \end{aligned} \tag{32}$$

$$\begin{aligned} |\dot{y}(s)-\dot{y}^m(s)| &= |<y_i(.)-y_i^m(.),\frac{\partial}{\partial s}R_s^0(.)>_{W_2^{2,0}}| \\ &\le \nu_i^2\|y_i-y_i^m\|_{W_2^{2,0}}, \quad i=1,\dots,n, \end{aligned} \tag{33}$$

and

$$\begin{aligned} |z_i(s)-z_i^m(s)| &= |<z_i(.)-z_i^m(.),R_s^1(.)>_{W_2^{2,1}}| \\ &\le \theta_i^1\|z_i-z_i^m\|_{W_2^{2,1}}, \quad i=1,\dots,n, \end{aligned} \tag{34}$$

$$\begin{aligned} |\dot{z}(s)-\dot{z}^m(s)| &= |<z_i(.)-z_i^m(.),\frac{\partial}{\partial s}R_s^1(.)>_{W_2^{2,1}}| \\ &\le \theta_i^2\|z_i-z_i^m\|_{W_2^{2,1}}, \quad i=1,\dots,n, \end{aligned} \tag{35}$$

where ν_i^1, ν_i^2, θ_i^1, and θ_i^2 are real constants.
Hence, we deduce that $\|y-y^m\|\to 0$, $\|z-z^m\|\to 0$, $\|\dot{y}-\dot{y}^m\|\to 0$ and $\|\dot{z}-\dot{z}^m\|\to 0$ as $m\to\infty$, where $\|y\|^2=\sum_{i=1}^n\|y_i\|_\infty^2$ and $\|z\|^2=\sum_{i=1}^n\|z_i\|_\infty^2$. Thus the proof is completed. □

4.1.2 Error analysis

In the following, we obtain the error estimates for the approximate solution of (22) in $\widetilde{W}[0,\mathrm{T}]$.

Theorem 7. Let the partition for the intervel $[0,1]$ be denoted by $P_m=\{0=t_1<t_2<\dots<t_m=\mathrm{T}\}$, also, suppose that $(y(t),z(t))$ and the problem (22) has an approximate solution $(y^m(t),z^m(t))$ in the space $\widetilde{W}[0,\mathrm{T}]$. The following relation holds,

$$\|(y(t),z(t))-(y^m(t),z^m(t))\|\le c\ h_t, \quad h_t=\max_{1\le i\le m-1}(t_{i+1}-t_i), \tag{36}$$

where c is a real constant and

$$\|(y(t),z(t))-(y^m(t),z^m(t))\|^2=\sum_{i=1}^n\|y_i-y_i^m\|_\infty+\sum_{i=1}^n\|z_i-z_i^m\|_\infty.$$

Proof. Assume that $t \in [t_i, t_{i+1}]$, for some $i = 1, ..., m-1$. We can write

$$\begin{aligned}(y(t), z(t)) - (y^m(t), z^m(t)) &= (y(t), z(t)) - (y(t_i), z(t_i)) \\ &+ (y^m(t_i), z^m(t_i)) - (y^m(t), z^m(t)) \\ &+ (y(t_i), z(t_i)) - (y^m(t_i), z^m(t_i)). \qquad (37)\end{aligned}$$

According to the mean value theorem, there exists $\xi_i \in (t_i, t_{i+1})$ such that

$$(y(t), z(t)) - (y(t_i), z(t_i)) = (t - t_i)(\dot{y}(\xi_i), \dot{z}(\xi_i)), \qquad (38)$$

where $\dot{y} = (\dot{y}_1, ..., \dot{y}_n)^T$ and $\dot{z} = (\dot{z}_1, ..., \dot{z}_n)^T$.
Since $(y(t), z(t)) \in \widetilde{W}[0, \mathrm{T}]$ then for some $d > 0$

$$\|(\dot{y}(t), \dot{z}(t))\| \leq d, \ \ \forall t \in [t_i, t_{i+1}], \qquad (39)$$

and therefore

$$\|(y(t), z(t)) - (y(t_i), z(t_i))\| \leq d \ h_t. \qquad (40)$$

We know

$$\begin{cases} |y^m(t) - y^m(t_i)| \leq \int_{t_i}^{t} |\dot{y}^m(s)| ds, \\ \\ |z^m(t) - z^m(t_i)| \leq \int_{t_i}^{t} |\dot{z}^m(s)| ds, \end{cases} \qquad (41)$$

where

$$\int_{t_i}^{t} |\dot{y}^m(s)| ds = (\int_{t_i}^{t} |\dot{y}_1^m(s)| ds, ..., \int_{t_i}^{t} |\dot{y}_n^m(s)| ds)^T,$$

and

$$int_{t_i}^{t} |\dot{z}^m(s)| ds = (\int_{t_i}^{t} |\dot{z}_1^m(s)| ds, ..., \int_{t_i}^{t} |\dot{z}_n^m(s)| ds)^T.$$

Since $(y^m(t), z^m(t)) \in \widetilde{W}[0, \mathrm{T}]$, it follows that

$$\|(y^m(t), z^m(t)) - (y^m(t_i), z^m(t_i))\| \leq e \ h_t, \qquad (42)$$

where e is a positive constant.
Using Theorem 6, for large m we have

$$\|(y(t_i), z(t_i)) - (y^m(t_i), z^m(t_i))\| \leq \epsilon. \qquad (43)$$

Since ϵ is arbitrary, we can combine (37)-(43), for the chosen value of m, thus

$$\|(y(t), z(t)) - (y^m(t), z^m(t))\| \leq c \ h_t, \quad h_t = \max_{1 \leq i \leq m-1} (t_{i+1} - t_i), \qquad (44)$$

where c is a positive constant and this completes the proof. □

Theorem 8. Consider the partition of the interval $[0, 1]$, denoted by $P_m = \{0 = t_1 < t_2 < ... < t_m = \mathrm{T}\}$, also, suppose that the problem (22) has an approximate solution $(y^m(t), z^m(t))$ in the space $\widetilde{W}[0, \mathrm{T}]$ such that $\|\ddot{y}^m(t)\|_\infty$ and $\|\ddot{z}^m(t)\|_\infty$ are bounded. If $(y(t), z(t)) \in \bigoplus_{i=1}^{2n} C^2[0, \mathrm{T}]$, then the following relations hold,

$$\|(y(t), z(t)) - (y^m(t), z^m(t))\| \leq c\ h_t^2, \tag{45}$$

$$\|(\dot{y}(t), \dot{z}(t)) - (\dot{y}^m(t), \dot{z}^m(t))\| \leq e\ h_t, \quad h_t = \max_{1\leq i\leq m-1}(t_{i+1} - t_i), \tag{46}$$

where c and e are real constants.

Proof. In each subinterval $[t_i, t_{i+1}]$, we can write

$$\begin{aligned}(\dot{y}(t), \dot{z}(t)) - (\dot{y}^m(t), \dot{z}^m(t)) \quad &= \quad (\dot{y}(t), \dot{z}(t)) - (\dot{y}(t_i), \dot{z}(t_i)) \\ &+ \quad (\dot{y}^m(t_i), \dot{z}^m(t_i)) - (\dot{y}^m(t), \dot{z}^m(t)) \\ &+ \quad (\dot{y}(t_i), \dot{z}(t_i)) - (\dot{y}^m(t_i), \dot{z}^m(t_i)).\end{aligned} \tag{47}$$

According to the mean value theorem, there exists $\xi_i \in (t_i, t_{i+1})$ such that

$$(\dot{y}(t), \dot{z}(t)) - (\dot{y}(t_i), \dot{z}(t_i)) = (t - t_i)(\ddot{y}(\xi_i), \ddot{z}(\xi_i)). \tag{48}$$

Since $(y(t), z(t)) \in \bigoplus_{i=1}^{2n} C^2[0, \mathrm{T}]$ then for some $d > 0$

$$\|(\ddot{y}(t), \ddot{z}(t))\| \leq d, \ \forall t \in [0, \mathrm{T}], \tag{49}$$

and therefore

$$\|(\dot{y}(t), \dot{z}(t)) - (\dot{y}(t_i), \dot{z}(t_i))\| \leq d\ h_t. \tag{50}$$

Note that

$$\begin{cases} |\dot{y}^m(t) - \dot{y}^m(t_i)| \leq \int_{t_i}^t |\ddot{y}^m(s)|ds, \\ \\ |\dot{z}^m(t) - \dot{z}^m(t_i)| \leq \int_{t_i}^t |\ddot{z}^m(s)|ds, \end{cases} \tag{51}$$

where

$$\int_{t_i}^t |\ddot{y}^m(s)|ds = (\int_{t_i}^t |\ddot{y}_1^m(s)|ds, \ldots, \int_{t_i}^t |\ddot{y}_n^m(s)|ds)^T,$$

and

$$\int_{t_i}^t |\ddot{z}^m(s)|ds = (\int_{t_i}^t |\ddot{z}_1^m(s)|ds, \ldots, \int_{t_i}^t |\ddot{z}_n^m(s)|ds)^T.$$

Hence

$$\|(\dot{y}^m(t), \dot{z}^m(t)) - (\dot{y}^m(t_i), \dot{z}^m(t_i))\| \leq k\ h_t. \tag{52}$$

Using Theorem 6 for large m, we have

$$\|(y(t_i), z(t_i)) - (y^m(t_i), z^m(t_i))\| \leq \epsilon, \tag{53}$$

and

$$\|(\dot{y}(t_i), \dot{z}(t_i)) - (\dot{y}^m(t_i), \dot{z}^m(t_i))\| \leq \epsilon. \tag{54}$$

Since ϵ is arbitrary, we can combine the equations (47)-(54), for the chosen value of m, thus

$$\|(\dot{y}(t),\dot{z}(t))-(\dot{y}^m(t),\dot{z}^m(t))\| \le d\ h_t, \quad h_t = \max_{1\le i\le m-1}(t_{i+1}-t_i). \tag{55}$$

We know that

$$y(t)-y^m(t) = y(t_i)-y^m(t_i)+\int_{t_i}^{t}(\dot{y}(s)-\dot{y}^m(s))ds, \tag{56}$$

$$z(t)-z^m(t) = z(t_i)-z^m(t_i)+\int_{t_i}^{t}(\dot{z}(s)-\dot{z}^m(s))ds. \tag{57}$$

By using (53)- (57) for large m, it is straightforward to see that

$$\|(y(t),z(t))-(y^m(t),z^m(t))\| \le c\ h_t^2, \quad h_t = \max_{1\le i\le m-1}(t_{i+1}-t_i), \tag{58}$$

and this completes the proof. □

4.2 The Nonlinear Problem

If (20) is nonlinear, then the approximate solution can be obtained using the following method.
The system $\{\rho_{ij}(t)\}_{(1,1)}^{(\infty,2n)}$ is complete in $\widetilde{W}[0,\mathrm{T}]$, then

$$(y(t),z(t)) = \sum_{i=1}^{\infty}\sum_{j=1}^{2n} C_{ij}\rho_{ij}(t). \tag{59}$$

Now, by the m-term intercept of (59), the approximate solution is presented by

$$Q_m(y(t),z(t)) = (y^m(t),z^m(t)) = \sum_{i=1}^{m}\sum_{j=1}^{2n} C_{ij}\rho_{ij}(t), \tag{60}$$

where $Q_m : \widetilde{W}[0,\mathrm{T}] \to \{\rho_{ij}(t)\}_{(1,1)}^{(m,2n)}$ is an orthogonal projection operator.
For numerical computation, we give m and initial function $(y_0^m(t),z_0^m(t)) \in \widetilde{W}[0,\mathrm{T}]$ and suppose that

$$\begin{aligned}(y_l^m(t),z_l^m(t)) &= (y_{1,l}^m(t),...,y_{n,l}^m(t),z_{1,l}^m(t),...,z_{n,l}^m(t))^T \\ &= \sum_{i=1}^{m}\sum_{j=1}^{2n} C_{ij,l}\rho_{ij}(t), \ l=1,2,\ldots, \end{aligned} \tag{61}$$

where the coefficients $C_{ij,l}, i=1,...,m, \ j=1,...,2n, \ l=1,2,...$ can be obtained by using

$$\Lambda(y_l^m(t_k),z_l^m(t_k)) = (F_0(t_k)+F_N(t_k,y_{l-1}^m(t_k)),G_0(t_k)+G_N(t_k,y_{l-1}^m(t_k),z_{l-1}^m(t_k)), \tag{62}$$

for $k=1,\ldots,m, \ l=1,2\ldots$.

4.2.1 The existence of solution and convergence analysis

In the following lemma, we find the solution of equation (20), and then show that the sequnce $\{(y_l^m(t), z_l^m(t))\}_{l=1}^{\infty}$ is convergent.

Lemma 3. (see [28]) For any $y_i \in W_2^{2,0}$, $i = 1, 2, ..., n$, and $z_i \in W_2^{2,1}$, $i = 1, 2, ..., n$, we have the following statements

$$\|y_i(t)\|_\infty \le \alpha_i^1 \ \|y_i(t)\|_{W_2^{2,0}}, \ \|\dot{y}_i(t)\|_\infty \le \alpha_i^2 \ \|y_i(t)\|_{W_2^{2,0}}, \tag{63}$$

$$\|z_i(t)\|_\infty \le \beta_i^1 \ \|z_i(t)\|_{W_2^{2,1}}, \ \|\dot{z}_i(t)\|_\infty \le \beta_i^2 \ \|z_i(t)\|_{W_2^{2,1}}, \tag{64}$$

where α_i^1, α_i^2, β_i^1 and β_i^2 are real constants.

Lemma 4. Suppose that, $\|y_i(t)\|_{W_2^{2,0}}$, $i = 1, ..., n$, and $\|z_i(t)\|_{W_2^{2,1}}$, $i = 1, ..., n$, are bounded, then there exist constants γ_i^1, γ_i^2, δ_i^1 and δ_i^2 such that

$$\|y_i(t)\|_\infty \le \gamma_i^1, \ \|\dot{y}_i(t)\|_\infty \le \gamma_i^2, \tag{65}$$

$$\|z_i(t)\|_\infty \le \delta_i^1, \ \|\dot{z}_i(t)\|_\infty \le \delta_i^2. \tag{66}$$

Proof. Since $\|y_i(t)\|_{W_2^{2,0}}$ and $\|z_i(t)\|_{W_2^{2,1}}$ are bounded, by Lemma 3, $\|y_i(t)\|_\infty$ and $\|z_i(t)\|_\infty$ are also bounded. □

Lemma 5. If γ_i, $i = 1, ..., n$, and δ_i, $i = 1, ..., n$, are real constants then $\Pi_i = \{y_{i,l}^m(t) | \ \|y_{i,l}^m(t)\|_{W_2^{2,0}} \le \gamma_i\} \subset C[0, \mathrm{T}]$, $i = 1, ..., n$, and $\widetilde{\Pi}_i = \{z_{i,l}^m(t) | \ \|z_{i,l}^m(t)\|_{W_2^{2,1}} \le \delta_i\} \subset C[0, \mathrm{T}]$, $i = 1, ..., n$, are bounded sets.

Proof. By Lemma 4, there exist positive constants $\gamma_i < \infty$, $i = 1, ..., n$, such that $\|y_{i,l}^m(t)\|_\infty \le \gamma_i$, for each $t \in [0, \mathrm{T}]$ and each $y_{i,l}^m(t) \in \Pi_i$. A similar argument shows that $\widetilde{\Pi}_i$, $i = 1, ..., n$ are bounded sets. □

Lemma 6. If γ_i, $i = 1, ..., n$, and δ_i, $i = 1, ..., n$, are real constants then $\Pi_i = \{y_{i,l}^m(t) | \ \|y_{i,l}^m(t)\|_{W_2^{2,0}} \le \gamma_i\} \subset C[0, \mathrm{T}]$, $i = 1, ..., n$, and $\widetilde{\Pi}_i = \{z_{i,l}^m(t) | \ \|z_{i,l}^m(t)\|_{W_2^{2,1}} \le \delta_i\} \subset C[0, \mathrm{T}]$, $i = 1, 2, ...n$, are equicontinuous.

Proof. Based on Lemma 5, for an arbitrary $y_{i,l}^m \in \Pi_i$, $i = 1, ..., n$, we deduce

$$\begin{aligned} |y_{i,l}^m(t') - y_{i,l}^m(t)| &= |\langle y_{i,l}^m(s), R_{t'}^0(s) - R_t^0(s)\rangle_{W_2^{2,0}}| \\ &\le \|y_{i,l}^m(t)\|_{W_2^{2,0}} \ \|R_{t'}^0(s) - R_t^0(s)\|_{W_2^{2,0}} \\ &\le \|y_{i,l}^m(t)\|_{W_2^{2,0}} \ \|\frac{d}{dt} R_t^0(s)|_{t=\zeta\in[t',t]}\|_{W_2^{2,0}} \ |t' - t| \\ &\le \omega_i |t' - t|, \end{aligned} \tag{67}$$

where ω_i, $i = 1, ..., n$, are positive constants.

Choosing

$$\delta_i = \frac{\epsilon}{\omega_i},$$

gives that for all $t, t^{'} \in [0,, \mathrm{T}]$ with $|t^{'} - t| < \delta_i$, we have

$$|y_{i,l}^m(t^{'}) - y_{i,l}^m(t)| < \epsilon, \tag{68}$$

hence Π_i, $i = 1, ..., n$ are equicontinuous sets.
A similar argument shows that $\widetilde{\Pi}_i$, $i = 1, ..., n$, are equicontinuous sets. □

Theorem 9. Suppose that the following statements are true.

(i) $\{t_i\}_{i=1}^{\infty}$ is a countable dense subset in the domain $[0, \mathrm{T}]$.

(ii) $\Pi_i = \{y_{i,l}^m(t) | \, \|y_{i,l}^m(t)\|_{W_2^{2,0}} \le \gamma_i\} \subset C[0, \mathrm{T}]$, $i = 1, ..., n$,
and $\widetilde{\Pi}_i = \{z_{i,l}^m(t) | \, \|z_{i,l}^m(t)\|_{W_2^{2,1}} \le \delta_i\} \subset C[0, \mathrm{T}]$, $i = 1, ..., n$.

(iii) $(l_{ij})_{2n \times 2n}$ is an invertible operator of $(y(t), z(t))$.

(iv) $F_N(t, y(t))$ and $G_N(t, y(t), z(t))$ are continuous as $t \in [0, \mathrm{T}]$.

Then, there exist subsequences $\{y_{l_p}^m(t)\}_{p=1}^{\infty} \subseteq \bigoplus_{i=1}^n \Pi_i$, and $\{z_{l_p}^m(t)\}_{p=1}^{\infty} \subseteq \bigoplus_{i=1}^n \widetilde{\Pi}_i$, which $\{(y_{l_p}^m(t), z_{l_p}^m(t))\}_{p=1}^{\infty}$, converges uniformly to $(y(t), z(t))$, as $m \to \infty, p \to \infty$, where

$$(y(t), z(t)) = {(l_{ij})_{2n \times 2n}}^{-1}(F_0(t) + F_N(t, y(t)), G_0(t) + G_N(t, y(t), z(t))).$$

Proof. Using (62), we have

$$\Lambda(y_l^m(t_k), z_l^m(t_k)) = (F_0(t_k) + F_N(t_k, y_{l-1}^m(t_k)), G_0(t_k) + G_N(t_k, y_{l-1}^m(t_k), z_{l-1}^m(t_k))), \tag{69}$$

for $k = 1, \ldots, m, \; l = 1, 2, \ldots$.

It follows from Lemma 5 that Π_i, $i = 1, \ldots, n$, are precompact sets. Then, any sequence in Π_i, has a subsequence such that uniformly convergent and the limit of subsequence belongs to Π_i. Hence, by this principle, we show that there exists a sequence $\{l_p\}_{p=1}^{\infty}$, such that subsequences $\{y_{l_p}^m(t)\}_{p=1}^{\infty}$ and $\{z_{l_p}^m(t)\}_{p=1}^{\infty}$, are uniformly convergent and $(y(t), z(t)) = \lim_{p \to \infty, \; m \to \infty} (y_{l_p}^m(t), z_{l_p}^m(t))$. Using (69), we also have

$$\Lambda(y_{l_p}^m(t_k), z_{l_p}^m(t_k)) = (F_0(t_k) + F_N(t_k, y_{l_p-1}^m(t_k)), G_0(t_k) + G_N(t_k, y_{l_p-1}^m(t_k) z_{l_p-1}^m(t_k)), \tag{70}$$

for $k = 1, \ldots, m, \; l = 1, 2, \ldots$.

Since Λ, F_N and G_N are continuous and $\{t_i\}_{i=1}^{\infty}$ is dense in $[0, \mathrm{T}]$, after taking limits from both sides of (70), we have

$$m \to \infty, p \to \infty \Rightarrow \Lambda(y(t), z(t)) = (F_0(t) + F_N(t, y(t)), G_0(t) + G_N(t, y(t), z(t))).$$

It follows that

$$(y(t), z(t)) = \Lambda^{-1}(F_0(t) + F_N(t, y(t)), G_0(t) + G_N(t, y(t), z(t))),$$

from the existence of Λ^{-1}.
This completes the proof of the theorem. □

Theorem 10. Suppose that the assumptions of Theorem 9 are valid, then there exist subsequences $\{y_{l_p}^m(t)\}_{p=1}^{\infty} \subseteq \bigoplus_{i=1}^n \Pi_i$, and $\{z_{l_p}^m(t)\}_{p=1}^{\infty} \subseteq \bigoplus_{i=1}^n \widetilde{\Pi}_i$, in which $\{(\dot{y}_{l_p}^m(t), \dot{z}_{l_p}^m(t))\}_{p=1}^{\infty}$ converges uniformly to $(\dot{y}(t), \dot{z}(t))$, as $m \to \infty, p \to \infty$, where

$$(y(t), z(t)) = \Lambda^{-1}(F_0(t) + F_N(t, y(t)), G_0(t) + G_N(t, y(t), z(t))).$$

Proof. By Theorem 9, there exist subsequences $\{y_{l_p}^m(t)\}_{p=1}^{\infty} \subseteq \bigoplus_{i=1}^n \Pi_i$, and $\{z_{l_p}^m(t)\}_{p=1}^{\infty} \subseteq \bigoplus_{i=1}^n \widetilde{\Pi}_i$, in which $\{(y_{l_p}^m(t), z_{l_p}^m(t))\}$ converges uniformly to $(y(t), z(t))$, as $m \to \infty, p \to \infty$, where

$$(y(t), z(t)) = \Lambda^{-1}(F_0(t) + F_N(t, y(t)), G_0(t) + G_N(t, y(t), z(t))).$$

It follows from Lemma 4 that the sequences $\{y_{i,l_p}^m(t)\}_{p=1}^{\infty}$, $i = 1, \ldots, n$, $\{\dot{y}_{i,l_p}^m(t)\}_{p=1}^{\infty}$, $i = 1, \ldots, n$, $\{z_{i,l_p}^m(t)\}_{p=1}^{\infty}$, $i = 1, \ldots, n$, and $\{\dot{z}_{i,l_p}^m(t)\}_{p=1}^{\infty}$, $i = 1, \ldots, n$, are bounded. Then, there exist subsequences $\{y_{i,l_{p_s}}^m(t)\}_{s=1}^{\infty}$, $i = 1, \ldots, n$, $\{\dot{y}_{i,l_{p_s}}^m(t)\}_{s=1}^{\infty}$, $i = 1, \ldots, n$, $\{z_{i,l_{p_s}}^m(t)\}_{s=1}^{\infty}$, $i = 1, \ldots, n$, and $\{\dot{z}_{i,l_{p_s}}^m(t)\}_{s=1}^{\infty}$, $i = 1, \ldots, n$, such that

$$\|y_{i,l_{p_s}}^m(t) - y_i(t)\|_{\infty} \to 0, \; \|\dot{y}_{i,l_{p_s}}^m(t) - \dot{y}_i(t)\|_{\infty} \to 0, \; as \; s \to \infty, \; m \to \infty, \; i = 1, \ldots, n,$$

$$\|z_{i,l_{p_s}}^m(t) - z_i(t)\|_{\infty} \to 0, \; \|\dot{z}_{i,l_{p_s}}^m(t) - \dot{z}_i(t)\|_{\infty} \to 0, \; as \; s \to \infty, \; m \to \infty, \; i = 1, \ldots, n.$$

Now, without loss of generality, we replace $\{y_{i,l_{p_s}}^m(t)\}_{s=1}^{\infty}$, $i = 1, \ldots, n$, and $\{z_{i,l_{p_s}}^m(t)\}_{s=1}^{\infty}$, $i = 1, \ldots, n$, with $\{y_{i,l_p}^m(t)\}_{p=1}^{\infty}$, $i = 1, \ldots, n$, and $\{z_{i,l_p}^m(t)\}_{p=1}^{\infty}$, $i = 1, \ldots, n$, respectively, which completes the proof of the theorem. □

Theorem 11. Suppose that the assumptions of Theorem 9 are valid, and there exists a unique solution for (20), then

$$\|(y_l^m(t), z_l^m(t)) - (y(t), z(t))\| \to 0, \; \text{as } l \to \infty, m \to \infty, \tag{71}$$
$$\|(\dot{y}_l^m(t), \dot{z}_l^m(t)) - (\dot{y}(t), \dot{z}(t))\| \to 0, \; \text{as } l \to \infty, m \to \infty. \tag{72}$$

Proof. Suppose that there exists $k \in \{1, 2, \ldots, n\}$ such that $\{y_{k,l}^m(t)\}_{l \geq 1} \subset \Pi_k$ does not converge to y_k. Then there exists a positive number $\epsilon_{k,0}$, and a subsequence $\{y_{k,l_p}^m(t)\}_{p \geq 1} \subset \Pi_k$, such that

$$\|y_{k,l_p}^m(t) - y_k(t)\|_{\infty} \geq \epsilon_{k,0}, \quad p = 1, 2, \ldots, \quad as \quad m \to \infty. \tag{73}$$

Since $\{y_{k,l}^m(t)\}_{l \geq 1} \subset \Pi_k$ is a precompact set, there exists a subsequence of $\{y_{k,l_p}^m(t)\}_{p \geq 1}$ which converges uniformly to $\widehat{y}_k(t)$. Without loss of generality, we may assume that $\{y_{k,l_p}^m(t)\}_{p \geq 1}$ converges uniformly to $\widehat{y}_k(t)$:

$$\|y_{k,l_p}^m(t) - \widehat{y}_k(t)\|_{\infty} \to 0, \; as \; p \to \infty, \; m \to \infty, \tag{74}$$

Since the solution of Equation (20) is unique, we have $y_k(t) = \widehat{y}_k(t)$, and so (74) contradicts (73). This completes the proof of Theorem 11. □

4.2.2 Error analysis

To demonstrate the error analysis, we present two following theorems.

Theorem 12. Suppose that the assumptions of Theorem 9 are valid and the partition of the interval $[0,1]$, denoted by $P_m = \{0 = t_1 < t_2 < ... < t_m = \mathrm{T}\}$, also suppose that $(y(t), z(t))$ and $(y_l^m(t), z_l^m(t))$ are respectively the exact solution and the approximate solution of the problem (20) in the space $\widetilde{W}[0,\mathrm{T}]$. The following relation holds,

$$\|(y(t), z(t)) - (y_l^m(t), z_l^m(t))\| \leq c\ h_t,\ as\ m \to \infty,\ l \to \infty, \tag{75}$$

where c is a real constant and $h_t = \max_{1 \leq i \leq m-1}(t_{i+1} - t_i)$.

Theorem 13. Suppose that the assumptions of Theorem 9 are valid and $(y^m(t), z^m(t))$ is the approximate solution of the problem (20) in the space $\widetilde{W}[0,\mathrm{T}]$ such that $\|\ddot{y}_l^m(t)\|$ and $\|\ddot{z}_l^m(t)\|$ are bounded and also suppose that the partition of the interval $[0,1]$, denoted by $P_m = \{0 = t_1 < t_2 < ... < t_m = \mathrm{T}\}$. If $(y(t), z(t)) \in \bigoplus_{i=1}^{2n} C^2[0,\mathrm{T}]$, then the following relations hold,

$$\|(y(t), z(t)) - (y_l^m(t), z_l^m(t))\| \leq c\ h_t^2,\quad as\ m \to \infty,\ l \to \infty, \tag{76}$$

$$\|(\dot{y}(t), \dot{z}(t)) - (\dot{y}_l^m(t), \dot{z}_l^m(t))\| \leq e\ h_t,\quad as\ m \to \infty,\ l \to \infty, \tag{77}$$

where c and e are real constants and $h_t = \max_{1 \leq i \leq m-1}(t_{i+1} - t_i)$.

5 Numerical Experiment

To indicate the performance of the current method, we present four examples. For each example, we choose the subset $t_i = \frac{i-1}{m-1},\ i = 1, ..., m$ to construct the orthogonal $\{\rho_{ij}(t)\}_{(1,1)}^{(m,2n)}$ in $\widetilde{W}$.
To illustrate the efficiency of the proposed method for Examples 1 and 3, we report error $|v(t) - v^m(t)|$ where $v^m(t) = -R^{-1}B(z^m(t) + p(\mathrm{T}))$ and for Examples 2 and 4, we report error $|v(t) - v_l^m(t)|$ where $v_l^m(t) = -R^{-1}B(z_l^m(t) + p(\mathrm{T}))$.
For the computational work we select the following examples from [10, 36, 32]. In the process of computation, all the calculations are done by using Maple 12 and Matlab 2012 software packages.

Example 1. The control problem is as follows:

$$\dot{x}(t) = v(t) - x(t),\quad 0 \leq t \leq 1, \tag{78}$$

where $x(0) = 1$. The quadratic objective functional to be minimized is as follows:

$$J[x(1), v(1)] = \frac{1}{2}\int_0^1 (x^2(t) + v^2(t))dt. \tag{79}$$

The exact solutions have been given in [10] by

$$x(t) = Ae^{\sqrt{2}t} + (1-A)e^{-\sqrt{2}t},$$
$$v(t) = A(\sqrt{2}+1)e^{\sqrt{2}t} - (1-A)(\sqrt{2}-1)e^{-\sqrt{2}t},$$

where $A = \frac{(2\sqrt{2}-3)}{-(e^{\sqrt{2}})^2+2\sqrt{2}-3}$.

In Table 1, the values of $v^m(t)$, by using the expressed method, for $m = 10, 50, 100, 150$ are compared with the exact solutions at selected values of time t. In Table 2, a comparison is made between the obtained results for $m = 10, 50, 100, 150$ together with the solutions obtained by Kafash et al. [24], Mehne et al. [26] for the optimal cost functional and the exact solution. In Figure 1, the values of $x(t)$ and $\dot{x}(t)$, by using the expressed method for $m = 150$ are compared with the exact solutions.

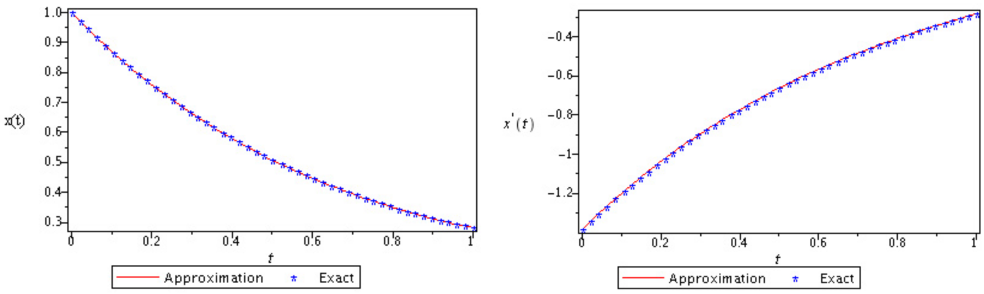

Figure 1: Estimated and exact values of $x(t)$ and $\dot{x}(t)$ for $m = 150$ (Example 1)

Table 1: Numerical comparison between the values of $v^m(t)$ for $m = 10, 50, 100, 150$ and the exact solutions at selected values of time t (Example 1)

t	$m = 10$	$m = 50$	$m = 100$	$m = 150$	*Exact*
0.0	−0.38495473	−0.38578941	−0.38581144	−0.38581543	−0.38581859
0.1	−0.32730314	−0.32803424	−0.32805378	−0.32805733	−0.32806014
0.2	−0.27621086	−0.27685108	−0.27686825	−0.27687137	−0.27687384
0.3	−0.23065788	−0.23121453	−0.23122941	−0.23123211	−0.23123424
0.4	−0.18973342	−0.19021029	−0.19022294	−0.19022523	−0.19022705
0.5	−0.15261816	−0.15301687	−0.15302734	−0.15302924	−0.15303074
0.6	−0.11856830	−0.11888911	−0.11889745	−0.11889895	−0.11890015
0.7	−0.08690088	−0.08714327	−0.08714951	−0.08715063	−0.08715152
0.8	−0.05698034	−0.05714335	−0.05714750	−0.05714825	−0.05714883
0.9	−0.02820598	−0.02828829	−0.02829036	−0.02829074	−0.02829104
1.0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000

Table 2: The optimal cost functional J (Example 1)

Methods	J	$Error$	$CPU-time\ (s)$
Method of Kafash et al. [24]			
Iteration $n=1$	0.194298642	$1.3e-3$	
Iteration $n=2$	0.192931607	$2.2e-5$	
Iteration $n=3$	0.192909776	$4.7e-7$	
Method of Mehne et al. [26]			
Iteration $n=1$	0.251362736	$5.8e-2$	
Iteration $n=2$	0.194298642	$1.3e-3$	
Iteration $n=3$	0.193828723	$9.1e-4$	
The proposed method			
$m=10$	0.19267632	$2.3e-4$	1.82 (s)
$m=50$	0.19290141	$7.9e-6$	7.11 (s)
$m=100$	0.19290737	$1.9e-6$	19.14 (s)
$m=150$	0.19290844	$8.5e-7$	58.42 (s)

Table 3: Numerical comparison between the values of $v^m(t)$ for $m = 200, 220, 240, 260$, $l = 25$ and the exact solutions at selected values of time t Estimated values of $v(t)$ for $m = 200, 220, 240, 260$ (Example 2)

t	$m=200$	$m=220$	$m=240$	$m=260$	$bvp4c$
0.0	−0.59775621	−0.59789871	−0.59834627	−0.59850812	−0.59869334
0.1	−0.56028712	−0.56037123	−0.56054103	−0.56056310	−0.56085005
0.2	−0.51860934	−0.51870638	−0.51884131	−0.51887531	−0.51888020
0.3	−0.47214962	−0.47217091	−0.47225834	−0.47230698	−0.47230918
0.4	−0.42035982	−0.42058920	−0.42067862	−0.42070901	−0.42071582
0.5	−0.36372467	−0.36373451	−0.36374529	−0.36375687	−0.36376888
0.6	−0.30124533	−0.30125321	−0.30126587	−0.30126809	−0.30127343
0.7	−0.23320118	−0.23320672	−0.23321234	−0.23322998	−0.23322692
0.8	−0.15984387	−0.15985391	−0.15986524	−0.15987193	−0.15988291
0.9	−0.08181139	−0.08181389	−0.08181512	−0.08181734	−0.08181862
1.0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000

Example 2. Consider the OCP

$$J[x_1(1), x_2(1), v(1)] = \frac{1}{2}\int_0^1 (x_1^2(t) + x_2^2(t) + v^2(t))dt, \tag{80}$$

provided with

$$\begin{cases} \dot{x}_1(t) = x_2(t) + x_1(t)x_2(t), \\ \dot{x}_2(t) = -x_1(t) + x_2(t) + x_2^2(t) + v(t), \\ x_1(0) = -0.8, \ x_2(0) = 0. \end{cases} \tag{81}$$

Table 3 presents the values of $v_l^m(t)$ using the proposed method for $m = 200, 220, 240, 260$, $l = 25$, and the solutions obtained by the Matlab package bvp4c. The cost functional values at the different methods are listed in Table 4.

Table 4: The optimal cost functional J (Example 2)

$Methods$	J	$CPU-time\ (s)$
Method of Shirazian et al. [36]		
n=10	0.44488	–
Method of Saberi Nik et al. [32]		
n=10	0.44488	–
The proposed method		
$m=200, l=25$	0.44424381	870.09 (s)
$m=220, l=25$	0.44445346	896.98 (s)
$m=240, l=25$	0.44477691	908.23 (s)
$m=260, l=25$	0.44484805	968.95 (s)

Example 3. In the following example, there is only one control function, $v(t)$, and two state functions, $x_1(t), x_2(t)$, and we are concerned with the minimization of

$$J[x(1), v(1)] = \frac{1}{2}\int_0^1 (x(t)\begin{pmatrix} 0 & 0 \\ 0 & 2\pi \end{pmatrix} x(t) + \frac{\pi}{2}v^2(t))dt, \tag{82}$$

provided with

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} 0 & 0 \\ \frac{\pi}{2} & 0 \end{pmatrix} x(t) + \begin{pmatrix} \frac{\pi}{2} \\ 0 \end{pmatrix} v(t), \quad 0 \le t \le 1, \\ x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \end{cases} \tag{83}$$

where $x(t) = (x_1(t), x_1(t))^T$. The exact solutions have been given by

$$x_1(t) = \frac{\cos(\frac{\pi}{2}t)e^{2\pi-\frac{\pi}{2}t} - 3\sin(\frac{\pi}{2}t)e^{2\pi-\frac{\pi}{2}t} + \cos(\frac{\pi}{2}t)e^{\pi+\frac{\pi}{2}t} - \sin(\frac{\pi}{2}t)e^{\pi+\frac{\pi}{2}t}}{e^{\pi}+e^{2\pi}},$$
$$x_2(t) = \frac{2\sin(\frac{\pi}{2}t)e^{2\pi-\frac{\pi}{2}t} + \cos(\frac{\pi}{2}t)e^{2\pi-\frac{\pi}{2}t} + \cos(\frac{\pi}{2}t)e^{\pi+\frac{\pi}{2}t}}{e^{\pi}+e^{2\pi}},$$
$$v(t) = \frac{2\sin(\frac{\pi}{2}t)e^{2\pi-\frac{\pi}{2}t} - 4\cos(\frac{\pi}{2}t)e^{2\pi-\frac{\pi}{2}t} - 2\sin(\frac{\pi}{2}t)e^{\pi+\frac{\pi}{2}t}}{e^{\pi}+e^{2\pi}}.$$

Table 5 presents the approximation of $v^m(t)$ using the expressed method for $m = 60, 200, 220$ and 240, compared with the exact solution. In Table 6, we give the absolute errors for the optimal cost functional, taking $m = 60, 200, 220, 240$ which proves the accuracy of the solution. In Figure 2, the values of $x_i(t)$ and $\dot{x}_i(t)$, $i = 1, 2$ using the expressed method for $m = 200$ are compared with the exact solutions.

5.1 The Controlled Van der Pol oscillator

Example 4. Consider the optimal control of the Van der Pol oscillator as given in [33]: Minimize

Table 5: Numerical comparison between the values of $v^m(t)$ for $m = 65, 75, 85, 150$ and the exact solutions at selected values of time t (Example 3)

t	$m = 60$	$m = 200$	$m = 220$	$m = 240$	$Exact$
0.0	−3.83117212	−3.83398029	−3.83411316	−3.83416097	−3.83430467
0.1	−2.99300554	−2.99519508	−2.99529627	−2.99533269	−2.99544214
0.2	−2.26411644	−2.26567275	−2.26574640	−2.26577291	−2.26585257
0.3	−1.64835768	−1.64942895	−1.64947964	−1.64949788	−1.64955269
0.4	−1.14423126	−1.14491626	−1.14494867	−1.14496033	−1.14499538
0.5	−0.74613098	−0.74652552	−0.74654419	−0.74655090	−0.74657110
0.6	−0.44562472	−0.44581671	−0.44582580	−0.44582906	−0.44583890
0.7	−0.23242578	−0.23249247	−0.23249560	−0.23249674	−0.23250013
0.8	−0.09513159	−0.09513455	−0.09513469	−0.09513476	−0.09513492
0.9	−0.02174069	−0.02172789	−0.02172726	−0.02172704	−0.02172638
1.0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000

Table 6: The optimal cost functional J (Example 3)

m	J	$Error$	$CPU-time$ (s)
60	4.74390944	$7.5e-3$	35.53 (s)
200	4.75079233	$6.6e-4$	166.65 (s)
220	4.75106701	$3.9e-4$	215.47 (s)
240	4.75116701	$2.9e-4$	287.59 (s)

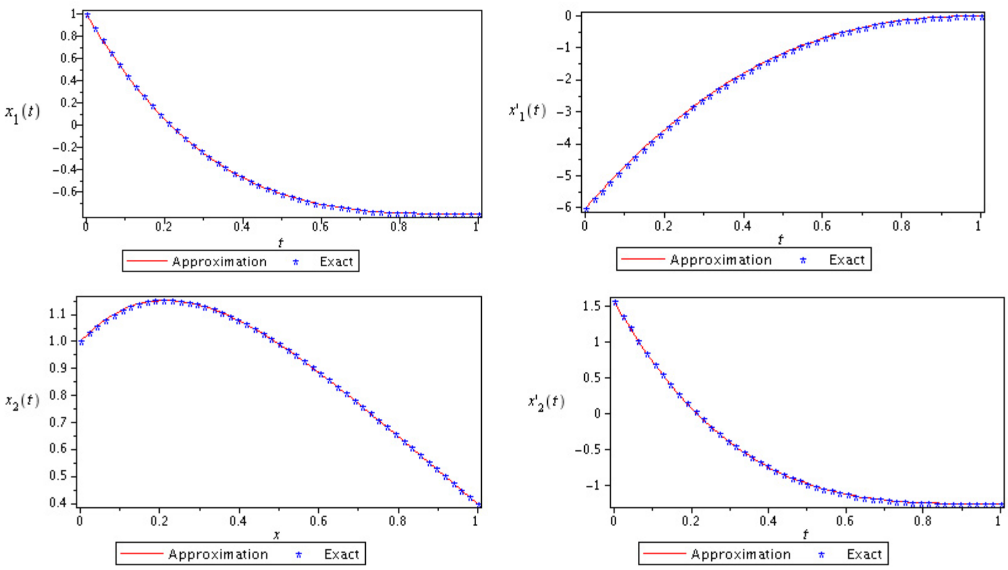

Figure 2: Estimated and exact values of $x_i(t), i = 1, 2$, and $\dot{x}_i(t), i = 1, 2$, for $m = 200$ (Example 3)

Table 7: Comparison of bvp4c, DTM and the expressed method for $v(t)$

t	$m = 400,\ l = 40$	DTM	$bvp4c$
0.0	0.01125972	−0.010732	−0.011266
0.4	0.51481165	0.51524	0.51484
0.8	0.72141451	0.72165	0.72142
1.2	0.64601920	0.64615	0.64604
1.6	0.37309510	0.37309	0.37310
2.0	−2.9298e − 7	− 1.5363e − 6	0.00000

Table 8: Numerical comparison of bvp4c, DTM and the proposed method

	$x_1(t)$			$x_2(t)$		
t	$m = 400,$ $l = 40$	DTM	$bvp4c$	$m = 400$ $l = 40$	DTM	$bvp4c$
0.0	0.99964932	0.99951	1.00000	−3.0347e − 9	−2.0587e − 7	0.000000
0.4	0.93585385	0.93584	0.93587	−0.28348142	−0.28344	−0.28349
0.8	0.79427145	0.79432	0.79430	−0.40884291	−0.40875	−0.40894
1.2	0.61307301	0.61321	0.61313	−0.50071973	−0.50061	−0.50093
1.6	0.38381050	0.38414	0.38390	−0.66461615	−0.66458	−0.66498
2.0	0.05994144	0.05927	0.05981	−0.98676550	−0.98677	−0.98678

$$J[x(2), v(2)] = \frac{1}{2}\int_0^2 (x^2(t) + \dot{x}^2(t) + v^2(t))dt, \tag{84}$$

provided with

$$\begin{cases} \ddot{x}(t) = \epsilon\omega(1 - x^2(t))\dot{x}(t) - \omega^2 x(t) + v(t),\ t \in [0, 2], \\ x(0) = 1,\ \dot{x}(0) = 0. \end{cases} \tag{85}$$

The OCP in Eqs. (84)-(85) may be restated as follows:
Minimize

$$J[x_1(2), x_2(2), v(2)] = \frac{1}{2}\int_0^2 (x_1^2(t) + x_2^2(t) + v^2(t))dt, \tag{86}$$

provided with

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = \epsilon\omega(1 - x_1^2(t))x_2(t) - \omega^2 x_1(t) + v(t),\ t \in [0, 2], \\ x_1(0) = 1,\ x_2(0) = 0. \end{cases} \tag{87}$$

Consider (87) with $\omega = 1$ and $\epsilon = 1$. Table 7 presents the approximation of $v_l^m(t)$ using the proposed method for $m = 400$, $l = 40$ compared with the results obtained using the Matlab package bvp4c and results given in [32] (DTM). In Table 8, a comparison is made between the values of $x_1(t)$ and $x_2(t)$ using the present method for $m = 400,\ l = 40$, together with the results obtained using the Matlab package bvp4c and results given in [32].

6 Conclusion

The objective of this paper was to propose a new semi-analytical technique to estimate the solution of quadratic OCPs. This method enabled us to solve the quadratic OCPs. The proposed method provided the solution in a convergent series with components that could be easily computed. Furthermore, the accuracy of the proposed technique was evaluated through numerical tests. The results from the numerical examples confirmed the accuracy and reliability of the analytical method for this equation.

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