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Research Article

On Grey Graphs and their Applications in Optimization

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Abstract. In this research, we use averages and relative measures of interval grey numbers to introduce *grey vertices*, *grey edges*, and *grey graphs* (*graphs are based on interval grey numbers*). To do so, we design a grey graph based on a *graph* (as the underlying graph). Also, we find a relation between grey vertices and grey edges of a grey graph. The primary method used in this research is based on linear inequalities related to grey vertices and grey edges. We find some necessary and sufficient conditions on the *grey vertex* (as *(non-)discrete grey vertices*) connectivity of grey graphs based on interval grey numbers and *linear inequality systems*. The paper includes implications for the development of *(non-)weighted graphs*, and the modeling of uncertainty problems by grey vertices, grey edges, and their relations in a *grey model* as a grey graph. As a weighted graph, a fuzzy graph is a vital graph that has some applications in the real world, but with changes in conditions, it loses its efficiency. On the other hand, the efficiency of a grey graph is stable under changes in the conditions. So, grey graphs cover the weaknesses of fuzzy graphs. The new conception of *grey graphs based on grey numbers* is introduced in this study. We propose an optimization method that can be applied for grey numbers in an extension of graphs, and apply it for gray numbers in the real world, especially for optimization problems and via gray graphs.

Keywords. (Interval) Grey number, Polar position, Optimization, Grey graph, Grey vertex, Grey edge.

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1 Introduction

As an essential approach to the study of uncertainty, *grey system theory* provides applied models for systems with both known and unknown data. Today, the theory has found many critical applications [6]. The introduction of grey systems was inspired by the *vagueness* of real-world data. Although fuzzy sets or rough sets have also been applied in the study of ambiguity or incompleteness of information, grey systems and fuzzy sets provide entirely different representations, and there is a fundamental difference between grey numbers and fuzzy subsets.

Recently, some researchers have studied grey systems and their subsystems known as *interval grey numbers*. Several types of interval grey numbers exist; for instance, according to Lin et al. [10, 11], a grey number is a number that belongs to a known range, but its exact value is unknown. Moreover, a grey number according to Yang et al. [18], is a number for which upper bound and lower bound are specified, whose exact position is unknown within the specified bounds.

In some other studies, interval grey numbers such as black and white numbers have been defined. Recall that a *black number* is a number for which one of the following conditions is satisfied.

- The number has neither an upper limit nor a lower limit.
- The upper limit and lower limit of the number are grey numbers.

Also, a number whose upper limit is equal to its lower limit is a *white number*.

Further details on grey numbers can be found in [1, 8, 12, 13, 19, 20]. To extend grey systems, some researchers studied grey (logical) algebraic systems and introduced some operations on grey algebraic systems and interval grey numbers. These include [3, 4, 5, 15, 16].

In general, complex systems are made of several interacting elements, and it is accordingly natural to associate a node to each element and a link to each interaction, which yields to a graph. As a result, graph theory is an important tool in modeling that enables us to utilize the rich information that lies in complex network structures, including food webs, scientific citations, social networks, communication networks, company links in a stock portfolio, the internet, and the world wide web. We refer the reader to [2] and [7] for instances of research carried out in this regard.

Another application of graph theory is in data structures, where data is transformed into information. For example, it is possible to use graph theory to provide a nonlinear representation of data in memory. Also, a graph and its adjacency matrix can represent the arbitrary relationships among data. Recently, graph theory has been combined with fuzzy subsets and neutrosophic subsets. The resulting techniques have found many applications in the real world. See [9] and [17] for more details.

In the present research, we introduce grey graphs as graphs that are supported by interval grey numbers. Indeed, we use interval grey numbers to define the notions of grey vertex and grey edge. Also, we extend them to graphs based on the relations between grey vertices and grey edges. The main motivation of this system comes from fuzzy vertices and fuzzy edges but in the different form of an (interval-valued) fuzzy graph.

Since there is a fundamental difference between grey numbers and fuzzy subsets, we find that graphs based on interval grey numbers are different from (interval-valued) fuzzy graphs. We try to find some results for equivalence conditions of connected vertices and one-valued edges. The solutions of linear inequalities play a crucial role in our work on the relation between grey vertices and grey edges. So, we find some algorithms for graphs based on interval grey numbers. Section 2 is devoted to reviewing those results and definitions that will be needed in the following sections. In Section 3, we present the novel concept of a grey graph and prove some of its properties. Finally, in Section 4, we present some applications of our results in the real world.

2 Notations and Preliminaries

This section is devoted to a review of those results and definitions that will be needed in what follows.

Definition 1. [21] Let U be an arbitrary set. A fuzzy subset of U is a set $F = \{(x, \mu_F(x)) : x \in U\}$, where $\mu_F : U \to [0,1]$ as the membership function of F and $\mu_F(x)$ represents the grade of belongingness of x into F.

Definition 2. [14] Assume V is an arbitrary set, $x, y \in V$, $\sigma : V \to [0,1]$ and $\mu : V \times V \to [0,1]$, where $\mu(x,y) \leq \sigma(x) \wedge \sigma(y)$. Then, $G = (V, \sigma, \mu)$ is a fuzzy graph, σ is as the fuzzy vertex set, and μ is as the fuzzy edge set of G.

Definition 3. [3] Consider a universal set U, $\Omega \subseteq \mathbb{R}$ and $a \in U$. A grey number is a number for which upper and lower bounds are specified, but whose exact position is unknown within the specified bounds. A discrete grey number is one for which a countable number of potential values is known. On the other hand, a continuous grey number is one which has the potential to take any value within an interval. A grey number expressed as $(a^{\pm} \in [a^-, a^+], a^- \leq a^+)$ is known as an interval grey number, where $[a^-, a^+] = \{s \mid a^- \leq s \leq a^+\}$, t is an information, $a^{\pm}$ is a grey number and, a^- and a^+ denote the lower and upper limits of the information. If $a^{\pm} \in [a^-, +\infty)$, we call it a lower-limit grey number, we refer to $a^{\pm} \in (-\infty, a^+]$ as an upper-limit grey number, and we call $a^{\pm} \in (-\infty, +\infty)$ a black number (that is, a number for which neither the exact value nor the range is known). Finally, a one-valued number a is one for which $a^- = a^+$ (that is, it is an exact value).

3 Graphs Based on Grey Numbers

The goal of this section is to define the notions of grey number-based vertex set (or grey vertex), grey number-based edge set (or grey edge), and grey number-based graph.

From now on, we consider $T_{min}(x, y) = \min\{x, y\}$ and $T_{pr}(x, y) = xy$, where $x, y \in \Omega \subseteq \mathbb{R}$.

Definition 4. Assume $\Omega = [a, b]$ is a universe, $\mu(\Omega) = b - a$ and $G^{\pm} = (V, E)$ is a graph. Given $xy \in E$, define $\mu^{-}(xy) = T_{min}(g^{\circ}(x^{\pm}), g^{\circ}(y^{\pm}))$ and $\mu^{+}(xy) = T_{min}(|Av(x^{\pm})|, |Av(y^{\pm})|)$, where $Av(x^{\pm}) = \dfrac{x^{+} + x^{-}}{2}$ and $g^{\circ}(x^{\pm}) = \dfrac{x^{+} - x^{-}}{b - a}$ denote the *average* and *relative measure* of the *grey number* $x^{\pm}$, respectively. Take

$$\sigma^{\pm} = \{x^{\pm} \in [\sigma^{-}(x), \sigma^{+}(x)] : \ x \in V\}, \mu^{\pm} = \{(xy)^{\pm} \in [\mu^{-}(xy), \mu^{+}(xy)] : \ xy \in E\}$$

and for all $x, y \in V$, $\ x^{\pm}, y^{\pm}$ and $(xy)^{\pm}$ are *grey numbers.*

An algebraic structure $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ is a *grey number-based graph* on $G^{\pm}$ (or more briefly, a *grey graph*), if for all $xy \in E$, $\mu^{-}(xy) \leq \mu^{+}(xy)$. If $\mu^{-}(xy) = \mu^{+}(xy) = 0$, we say that no link exists between the vertices $x^{\pm}$ and $y^{\pm}$. We call $\sigma^{\pm}$ the set of *grey number-based vertices* or *grey vertices*, and we refer to $\mu^{\pm}$ as the set of *grey number-based edges* or *grey edges* of G.

In what follows, we present an example of a grey number-based graph.

Example 1. Let $\Omega = [0, 1000]$ and consider the cycle graph C_4. Then, $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ is a grey number-based graph on C_4, depicted in Figure 1. Computations show that

$$g^{\circ}(a^{\pm}) = \frac{2-1}{1000}, g^{\circ}(b^{\pm}) = \frac{7-2}{1000}, |Av(a^{\pm})| = \frac{1+2}{2} \text{ and } |Av(b^{\pm})| = \frac{7+2}{2}.$$

Thus,

$$\begin{aligned} \mu^{-}(ab) &= T_{min}(g^{\circ}(a^{\pm}), g^{\circ}(b^{\pm})) \\ &= T_{min}(\frac{1}{1000}, \frac{5}{1000}) \\ &= \frac{1}{1000} \mu^{+}(ab) \\ &= T_{min}(|Av(a^{\pm})|, |Av(b^{\pm})|) \\ &= T_{min}(\frac{3}{2}, \frac{9}{2}) \\ &= \frac{3}{2}. \end{aligned}$$

Therefore,

$$(ab)^{\pm} \in [\mu^{-}(ab), \mu^{+}(ab)] = [\frac{1}{1000}, \frac{3}{2}].$$

By a similar argument,

$$(ac)^{\pm} \in [\mu^{-}(ac), \mu^{+}(ac)] = [\frac{1}{1000}, \frac{3}{2}], (bd)^{\pm} \in [\mu^{-}(bd), \mu^{+}(bd)] = [\frac{1}{1000}, \frac{7}{2}],$$

and

$$(cd)^{\pm} \in [\mu^{-}(cd), \mu^{+}(cd)] = [\frac{4}{1000}, \frac{9}{2}].$$

From now on, let Ω be a universe and consider a graph $G^{\pm} = (V, E)$. Also, let $xy \in E$ and $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ be a grey graph.

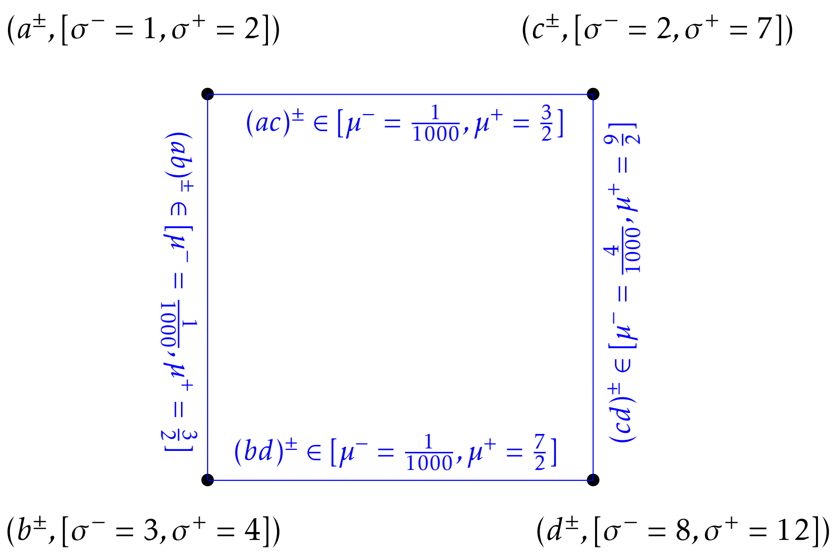

Figure 1: The grey graph $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ on C_4.

Theorem 1. *The following statements are true.*

(i) *If* $g^{\circ}(x^{\pm}) > |Av(x^{\pm})|$, *then* $0 \leq |Av(x^{\pm})| \leq 1$.

(ii) $\mu(\Omega) < \dfrac{\mu(x^{\pm})}{|Av(x^{\pm})|} \Leftrightarrow g^{\circ}(x^{\pm}) > |Av(x^{\pm})|$.

(iii) $\Big(|x^-| < x^+ \text{ and } g^{\circ}(x^{\pm}) > Av(x^{\pm})\Big) \Leftrightarrow \mu(\Omega) < \dfrac{\mu(x^{\pm})}{Av(x^{\pm})}$.

(iv) *If* $|Av(x^{\pm})| \geq 1$, *then* $g^{\circ}(x^{\pm}) \leq |Av(x^{\pm})|$.

Proof. Parts (i), (ii) and (iii) can be easily proved.
Also, (iv) can be deduced from (i). □

Theorem 2. *If* $\mu^{\pm}$ *is one-valued, then one of the following statements is true.*

(i) $g^{\circ}(x^{\pm}) \leq g^{\circ}(y^{\pm})$ *and* $|Av(x^{\pm})| \leq |Av(y^{\pm})|$.

(ii) $g^{\circ}(y^{\pm}) \leq g^{\circ}(x^{\pm})$ *and* $|Av(x^{\pm})| \leq |Av(y^{\pm})|$.

(iii) $g^{\circ}(x^{\pm}) \leq g^{\circ}(y^{\pm})$ *and* $|Av(y^{\pm})| \leq |Av(x^{\pm})|$.

(iv) $g^{\circ}(y^{\pm}) \leq g^{\circ}(x^{\pm})$ *and* $|Av(y^{\pm})| \leq |Av(x^{\pm})|$.

Proof. Let $xy \in E$. Since $\mu^{\pm}$ is one-valued, there is $t \in \mathbb{R}^+$ that $[\mu^-(xy), \mu^+(xy)] = \{t\}$. Let us denote this set by t for the sake of simplicity. Thus, $\mu^-(xy) = t = \mu^+(xy)$.

Now, $\mu^-(xy) = t$ implies $T_{min}(g^{\circ}(x^{\pm}), g^{\circ}(y^{\pm})) = t$, and so $g^{\circ}(x^{\pm}) = t < g^{\circ}(y^{\pm})$, $g^{\circ}(y^{\pm}) = t < g^{\circ}(x^{\pm})$ or $g^{\circ}(x^{\pm}) = t = g^{\circ}(y^{\pm})$. In addition, $\mu^+(xy) = t$ implies $T_{min}(|Av(x^{\pm})|, |Av(y^{\pm})|) = t$, and so $|Av(x^{\pm})| = t < |Av(y^{\pm})|$, $|Av(y^{\pm})| = t < |Av(x^{\pm})|$ or $|Av(y^{\pm})| = t = |Av(x^{\pm})|$.

Therefore,

$$\Big(g^{\circ}(x^{\pm}) = t \leq g^{\circ}(y^{\pm})\text{and}|Av(x^{\pm})| = t \leq |Av(y^{\pm})|\Big),$$
$$\Big(g^{\circ}(y^{\pm}) = t \leq g^{\circ}(x^{\pm})\text{and}|Av(x^{\pm})| = t \leq |Av(y^{\pm})|\Big),$$
$$\Big(g^{\circ}(x^{\pm}) = t \leq g^{\circ}(y^{\pm})\text{and}|Av(y^{\pm})| = t \leq |Av(x^{\pm})|\Big),$$

or

$$\Big(g^{\circ}(y^{\pm}) = t \leq g^{\circ}(x^{\pm})\text{and}|Av(y^{\pm})| = t \leq |Av(x^{\pm})|\Big).$$

□

In the following example, we show that the converse of Theorem 2 may not be true necessarily.

Example 2. Let $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ be the grey graph on graph C_3 that is depicted in Figure 2. Computations show that $g^{\circ}(a^{\pm}) \leq g^{\circ}(b^{\pm})$ and $|Av(a^{\pm})| \leq |Av(b^{\pm})|$, while $(ab)^{\pm}$ is not one-valued.

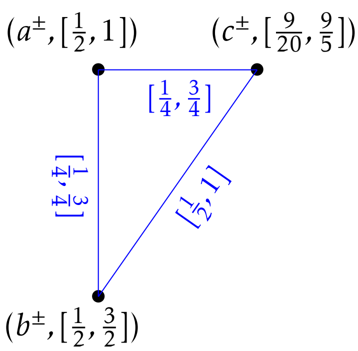

Figure 2: The grey graph $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ on C_3.

Definition 5. Let Ω be a universe and $x^{\pm} \subseteq \Omega$. Then, $x^{\pm}$ is *called* 1*-polar* if $T_{pr}(x^{-}, x^{+}) \geq 0$. Figures 3 and 4 show *(non-)negative-polar* and *(non-)positive-polar*, respectively.

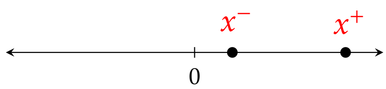

Figure 3: Position of the grey number $x^{\pm}$ on the real line ((non-)negative-polar).

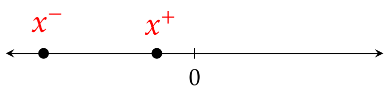

Figure 4: Position of the grey number $x^{\pm}$ on the real line ((non-)positive-polar).

Based on Theorem 2, we need to solve the linear inequalities $\Big(g^{\circ}(x^{\pm}) \leq g^{\circ}(y^{\pm})$ and $|Av(x^{\pm})| \leq |Av(y^{\pm})|\Big)$, $\Big(g^{\circ}(y^{\pm}) \leq g^{\circ}(x^{\pm})$ and $|Av(x^{\pm})| \leq |Av(y^{\pm})|\Big)$, $\Big(g^{\circ}(x^{\pm}) \leq g^{\circ}(y^{\pm})$ and $|Av(y^{\pm})| \leq |Av(x^{\pm})|\Big)$, and $\Big(g^{\circ}(y^{\pm}) \leq g^{\circ}(x^{\pm})$ and $|Av(y^{\pm})| \leq |Av(x^{\pm})|\Big)$.

We consider the three cases $\mu(\Omega) < 2,\ \mu(\Omega) = 2$ and $\mu(\Omega) > 2$.

Theorem 3. If $\mu(\Omega) = 2$ and $\mu^{\pm}$ is non-zero and one-valued, then one of the following statements is true.

(i) $\Big(x^+ = 0,\ y^- - y^+ \leq x^-$ and $-|y^- + y^+| \leq x^- \leq 0\Big)$ or $\Big(x^- = 0,\ x^+ \leq y^+ - y^-$ and $0 \leq x^+ \leq |y^+ + y^-|\Big)$.

(ii) $\Big(|x^+ + x^-| \leq |y^+ + y^-|$ and $x^- \leq 0\Big)$ or $\Big(y^+ + x^- - y^- \leq 0$ and $|x^+ + x^-| \leq |y^+ + y^-|\Big)$.

(iii) $\Big(y^- < 0$ and $|y^+ + y^-| \leq |x^+ + x^-|\Big)$ or $\Big(x^- - x^+ - y^- > 0$ and $|y^+ + y^-| \leq |x^+ + x^-|\Big)$.

(iv) $\Big(y^+ = 0,\ x^- - x^+ \leq y^-$ and $-|x^- + x^+| \leq y^- \leq 0\Big)$ or $\Big(y^- = 0$ and $y^+ \leq x^+ - x^-$ and $0 \leq y^+ \leq |x^+ + x^-|\Big)$.

Proof. Let $xy \in E$ and $t \in \mathbb{R}$. We use Theorem 2 in what follows.

(i) If $g^{\circ}(x^{\pm}) \leq g^{\circ}(y^{\pm})$ and $|Av(x^{\pm})| \leq |Av(y^{\pm})|$, then $g^{\circ}(x^{\pm}) = |Av(x^{\pm})|$. Since $\mu(\Omega) = 2$, $x^- = 0$ or $x^+ = 0$. Also, $g^{\circ}(x^{\pm}) \leq g^{\circ}(y^{\pm})$ and $|Av(x^{\pm})| \leq |Av(y^{\pm})|$ allow us to conclude that $x^+ \leq (y^+ + x^-) - y^-$ and $-(|y^+ + y^-| + x^-) \leq x^+ \leq (|y^+ + y^-|) - x^-$. Thus, for $\mu(\Omega) = 2$,

$$\Big(x^+ = 0,\ y^- - y^+ \leq x^- \text{and} - |y^- + y^+| \leq x^- \leq |y^+ + y^-|\Big)\text{or}$$
$$\Big(x^- = 0 \text{ and } x^+ \leq y^+ - y^- \text{and } 0 \leq x^+ \leq |y^+ + y^-|\Big).$$

(ii) If $\Big(g^{\circ}(y^{\pm}) \leq g^{\circ}(x^{\pm})$ and $|Av(x^{\pm})| \leq |Av(y^{\pm})|\Big)$, then $g^{\circ}(y^{\pm}) = |Av(x^{\pm})|$. Since $\mu(\Omega) = 2$, $x^+ = y^+ - (x^- + y^-)$ or $x^+ = y^- - (x^- + y^+)$. Also, $g^{\circ}(y^{\pm}) \leq g^{\circ}(x^{\pm})$ and $|Av(x^{\pm})| \leq |Av(y^{\pm})|$ allow us to conclude that $x^+ > (y^+ + x^-) - y^-$ and $-(|y^+ + y^-| + x^-) \leq x^+ \leq (|y^+ + y^-|) - x^-$. Thus, for $\mu(\Omega) = 2$,

$$\Big(|x^+ + x^-| \leq |y^+ + y^-| \text{ and } x^- \leq 0\Big)\text{or}$$
$$\Big(y^+ + x^- - y^- \leq 0 \text{ and } |x^+ + x^-| \leq |y^+ + y^-|\Big).$$

(iii) If $g^{\circ}(x^{\pm}) \leq g^{\circ}(y^{\pm})$ and $|Av(y^{\pm})| \leq |Av(x^{\pm})|$, then $g^{\circ}(x^{\pm}) = |Av(y^{\pm})|$. Since $\mu(\Omega) = 2$, $y^+ = x^+ - x^- - y^-$ or $y^+ = x^- - x^+ - y^-$. Also, $g^{\circ}(x^{\pm}) \leq g^{\circ}(y^{\pm})$ and $|Av(y^{\pm})| \leq |Av(x^{\pm})|$ allow us to conclude that $y^+ > (x^+ - x^-) + y^-$ and $-(|x^+ + x^-| + y^-) \leq y^+ \leq (|x^+ + x^-|) - y^-$. Thus, for $\mu(\Omega) = 2$,

$$\Big(y^- \leq 0 \text{ and} |y^+ + y^-| \leq |x^+ + x^-|\Big)\text{or}$$
$$\Big(x^- - x^+ - y^- > 0 \text{ and} |y^+ + y^-| \leq |x^+ + x^-|\Big).$$

(iv) If $g^{\circ}(y^{\pm}) \leq g^{\circ}(x^{\pm})$ and $|Av(y^{\pm})| \leq |Av(x^{\pm})|$, then $g^{\circ}(y^{\pm}) = |Av(y^{\pm})|$. Since $\mu(\Omega) = 2$, $y^- = 0$ or $y^+ = 0$. Also, $g^{\circ}(y^{\pm}) \leq g^{\circ}(x^{\pm})$ and $|Av(y^{\pm})| \leq |Av(x^{\pm})|$ allow us to conclude that $y^+ \leq (x^+ + y^-) - x^-$ and $-(|x^+ + x^-| + y^-) \leq y^+ \leq (|x^+ + x^-|) - y^-$. Thus, for $\mu(\Omega) = 2$,

$$\Big(y^+ = 0,\ x^- - x^+ \leq y^- \text{ and } - |x^- + x^+| \leq y^- \leq |x^+ + x^-|\Big)\text{or}$$
$$\Big(y^- = 0 \text{ and } y^+ \leq x^+ - x^- \text{ and } 0 \leq y^+ \leq |x^+ + x^-|\Big).$$

□

Example 3. (i) Let $\Omega = [-1,1]$ and consider the cycle graph C_3. Then, $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ is a grey graph on the graph C_3, depicted in Figure 5. Clearly, $\big(c^- = 0$ and $c^+ \leq a^+ - a^-$ and $0 \leq c^+ \leq |a^+ + a^-|\big)$, and $(ac)^{\pm}$ is one-valued.

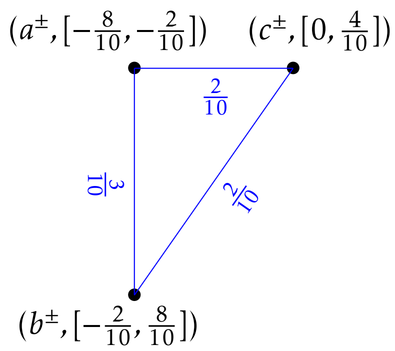

Figure 5: The grey graph $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ on C_3.

(ii) Let $\Omega = [-2,0]$ and consider the cycle graph C_3. Then, $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ is a grey graph on the graph C_3, depicted in Figure 6. Clearly, $b^- \leq 0$ and $a^+ - a^- \leq |a^+ + a^-|$, while $(ab)^{\pm}$ is not one-valued. This reveals that the converse of Theorem 3(iii) may not be true necessarily.

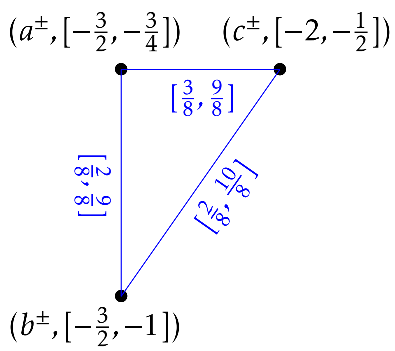

Figure 6: The grey graph $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ on C_3.

Theorem 4. *If $\mu(\Omega) = 2$ and one of the following statements is true, then $\mu^{\pm}$ is non-zero and one-valued.*

(i) $\big(x^+ = 0,\ y^- - y^+ \leq x^-$ *and* $-|y^- + y^+| \leq x^- \leq 0\big)$ *or* $\big(x^- = 0$ *and* $x^+ \leq y^+ - y^-$ *and* $0 \leq x^+ \leq |y^+ + y^-|\big)$.

(ii) $\big(x^+ = y^+ - (x^- + y^-)$ *and* $x^- \leq 0\big)$ *or* $\big(x^+ = y^- - (x^- + y^+), y^+ + x^- - y^- \leq 0$ *and (either* $y^{\pm}$ *is (non) positive-polar or is (non) negative-polar)*$\big)$.

(iii) $\big(y^+ = 0,\ x^- - x^+ \leq y^-$ *and* $-|x^- + x^+| \leq y^- \leq 0\big)$ *or* $\big(y^- = 0$ *and* $y^+ \leq x^+ - x^-$ *and* $0 \leq y^+ \leq |x^+ + x^-|\big)$.

Proof. The proof is similar to that of Theorem 3. □

Theorem 5. *If $0 < \mu(\Omega) < 2$ and $\mu^{\pm}$ is non-zero and one-valued, then one of the following statements is true.*

(i) $\left(x^{+} = (\frac{p}{q})x^{-}, x^{-} \leq \frac{q\mu(y^{\pm})}{2\mu(\Omega)} \text{ and } \frac{-q|Av(y^{\pm})|}{2} \leq x^{-} \leq \frac{q|Av(y^{\pm})|}{2}\right)$ *or*
$\left(x^{+} = (\frac{q}{p})x^{-}, x^{-} \geq \frac{-p\mu(y^{\pm}))}{2\mu(\Omega)} \text{ and } -\frac{p|Av(y^{\pm})|}{2} \leq x^{-} \leq \frac{p|Av(y^{\pm})|}{2}\right).$

(ii) $\left(y^{+} = (\frac{p}{q})y^{-}, y^{-} \leq \frac{q\mu(x^{\pm})}{2\mu(\Omega)} \text{ and } \frac{-q|Av(x^{\pm})|}{2} \leq y^{-} \leq \frac{q|Av(x^{\pm})|}{2}\right)$ *or*
$\left(y^{+} = (\frac{q}{p})y^{-}, y^{-} \geq \frac{(p)(-\mu(x^{\pm}))}{2\mu(\Omega)} \text{ and } \frac{-p|Av(x^{\pm})|}{2} \leq y^{-} \leq \frac{p|Av(x^{\pm})|}{2}\right).$

Here, $p = 2 + \mu(\Omega)$ and $q = 2 - \mu(\Omega)$.

Proof. Let $xy \in E$ and $t \in \mathbb{R}$. We use Theorem 2 in what follows.
(i) If $g^{\circ}(x^{\pm}) = t \leq g^{\circ}(y^{\pm})$ and $|Av(x^{\pm})| = t \leq |Av(y^{\pm})|$, then

$$g^{\circ}(x^{\pm}) = |Av(x^{\pm})|, g^{\circ}(x^{\pm}) \leq g^{\circ}(y^{\pm}) \text{ and } |Av(x^{\pm})| \leq |Av(y^{\pm})|.$$

Since $\mu(\Omega) \neq 2$, $g^{\circ}(x^{\pm}) = |Av(x^{\pm})|$ implies $x^{+} = (\frac{2+\mu(\Omega)}{2-\mu(\Omega)})x^{-}$ or $x^{+} = (\frac{2-\mu(\Omega)}{2+\mu(\Omega)})x^{-}$. Also, $g^{\circ}(x^{\pm}) \leq g^{\circ}(y^{\pm})$ and $|Av(x^{\pm})| \leq |Av(y^{\pm})|$ allow us to conclude that

$$x^{+} \leq (y^{+} + x^{-}) - y^{-} and - (|y^{+} + y^{-}| + x^{-}) \leq x^{+} \leq (|y^{+} + y^{-}|) - x^{-}.$$

Thus, for $\mu(\Omega) < 2$,

$$\begin{aligned}
&\Big(x^{+} = (\frac{2+\mu(\Omega)}{2-\mu(\Omega)})x^{-}, x^{-} \leq \frac{(2-\mu(\Omega))(y^{+} - y^{-})}{2\mu(\Omega)} && \text{and} \\
&\frac{(2-\mu(\Omega))(|y^{+} + y^{-}|)}{4} \leq x^{-} \leq \frac{(\mu(\Omega)-2)(|y^{+} + y^{-}|)}{4}\Big) && \text{or} \\
&\Big(x^{+} = (\frac{2-\mu(\Omega)}{2+\mu(\Omega)})x^{-}, x^{-} \geq \frac{(2+\mu(\Omega))(y^{-} - y^{+})}{2\mu(\Omega)} && \text{and} \\
&\frac{(2+\mu(\Omega))(|y^{+} + y^{-}|)}{-4} \leq x^{-} \leq \frac{(\mu(\Omega)+2)(|y^{+} + y^{-}|)}{4}\Big).
\end{aligned}$$

(ii) If $g^{\circ}(y^{\pm}) \leq g^{\circ}(x^{\pm})$ and $|Av(y^{\pm})| \leq |Av(x^{\pm})|$, then $g^{\circ}(y^{\pm}) = |Av(y^{\pm})|$. Since $\mu(\Omega) \neq 2$, $g^{\circ}(y^{\pm}) = |Av(y^{\pm})|$ implies $y^{+} = (\frac{2+\mu(\Omega)}{2-\mu(\Omega)})y^{-}$ or $y^{+} = (\frac{2-\mu(\Omega)}{2+\mu(\Omega)})y^{-}$. In addition, $g^{\circ}(y^{\pm}) \leq g^{\circ}(x^{\pm})$ and $|Av(y^{\pm})| \leq |Av(x^{\pm})|$ allow us to conclude that

$$y^{+} \leq (x^{+} + y^{-}) - x^{-} and - (|x^{+} + x^{-}| + y^{-}) \leq y^{+} \leq (|x^{+} + x^{-}|) - y^{-}.$$

Thus, for $\mu(\Omega) < 2$,

$$\Big(y^{+} = (\frac{2+\mu(\Omega)}{2-\mu(\Omega)})y^{-}, y^{-} \leq \frac{(2-\mu(\Omega))(x^{+} - x^{-})}{2\mu(\Omega)} \qquad \text{and}$$

$$\frac{(2-\mu(\Omega))(|x^+ + x^-|)}{-4} \le y^- \le \frac{(-\mu(\Omega)+2)(|x^+ + x^-|)}{4}\Big) \quad \text{or}$$
$$\Big(y^+ = (\frac{2-\mu(\Omega)}{2+\mu(\Omega)})y^-, y^- \ge \frac{(2+\mu(\Omega))(x^- - x^+)}{2\mu(\Omega)} \quad \text{and}$$
$$\frac{(2+\mu(\Omega))(|x^+ + x^-|)}{-4} \le y^- \le \frac{(\mu(\Omega)+2)(|x^+ + x^-|)}{4}\Big).$$

□

Theorem 6. If $\mu(\Omega) \neq 2$ and $\mu^\pm$ is non-zero and one-valued, then one of the following statements is true.

(*i*) $\Big(q\mu(y^\pm)) - 2\mu(\Omega)x^- \ge 0\Big)$ and $-\mu(\Omega)|Av(y^\pm)| \le (\mu(y^\pm)) \le \mu(\Omega)|Av(y^\pm)|\Big)$ or $\Big(-p\mu(y^\pm) - 2\mu(\Omega)x^- \ge 0\Big)$ and $-\mu(\Omega)|Av(y^\pm)| \le \mu(y^\pm) \le \mu(\Omega)|Av(y^\pm)|\Big)$.

(*ii*) $\Big(y^+ = (\frac{2\mu(x^\pm) - y^-\mu(\Omega)}{\mu(\Omega)}), q\mu(x^\pm) - 2\mu(\Omega)y^- \ge 0$ and $-\mu(\Omega)|Av(x^\pm)| \le \mu(x^\pm) \le \mu(\Omega)|Av(x^\pm)|$ or $\Big(y^+ = (\frac{-2\mu(x^\pm) - y^-\mu(\Omega)}{\mu(\Omega)}), p\mu(x^\pm) + 2\mu(\Omega)y^- \le 0$ and $-\mu(\Omega)|Av(x^\pm)| \le \mu(x^\pm) \le \mu(\Omega)|Av(x^\pm)|$.

Here, $q = 2 - \mu(\Omega)$ and $p = 2 + \mu(\Omega)$.

Proof. The proof is similar to that of Theorem 5. □

In the following example, we observe that the converse of Theorem 6 may not be true necessarily.

Example 4. Let $\Omega = [-2, 3]$ and consider the cycle graph C_3. Then, $G = (\Omega, \sigma^\pm, \mu^\pm)$ is a grey graph on the graph C_3, depicted in Figure 7.

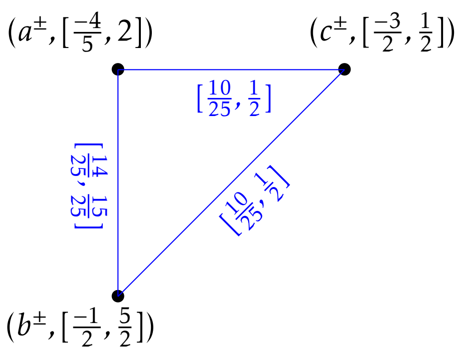

Figure 7: The grey graph $G = (\Omega, \sigma^\pm, \mu^\pm)$ on C_3.

Computations show that $(-3\mu(c^\pm)) - 10a^- > 0)$ and $-5|Av(c^\pm)| \le \mu(c^\pm) \le 5|Av(c^\pm)|$, while $(ca)^\pm$ is not one-valued.

Theorem 7. *Let Ω be a universe. Consider a graph $G^{\pm} = (V, E)$. Also, let $xy \in E$ and $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ be a grey graph with $\mu(\Omega) > 2$. Then, $\mu^{\pm}$ is non-zero and one-valued if and only if one of the following statements is true.*

(i) $\Big(x^{+} = (\frac{p}{q})x^{-}, x^{-} \geq \frac{q\mu(y^{\pm})}{2\mu(\Omega)}$ *and* $\frac{q|Av(y^{\pm})|}{2} \leq x^{-} \leq \frac{-q|Av(y^{\pm})|}{2}\Big)$ *or* $\Big(x^{+} = (\frac{q}{p})x^{-}, x^{-} \geq \frac{(p)(-\mu(y^{\pm}))}{2\mu(\Omega)}$ *and* $\frac{p|Av(y^{\pm})|}{-2} \leq x^{-} \leq \frac{p|Av(y^{\pm})|}{2}\Big)$.

(ii) $\Big(y^{+} = (\frac{p}{q})y^{-}, y^{-} \geq \frac{q\mu(x^{\pm})}{2\mu(\Omega)}$ *and* $\frac{q|Av(x^{\pm})|}{2} \leq y^{-} \leq \frac{-q|Av(x^{\pm})|}{2}\Big)$ *or* $\Big(y^{+} = (\frac{q}{p})y^{-}, y^{-} \geq \frac{(p)(-\mu(x^{\pm}))}{2\mu(\Omega)}$ *and* $\frac{p|Av(x^{\pm})|}{-2} \leq y^{-} \leq \frac{p|Av(x^{\pm})|}{2}\Big)$.

Here, $q = 2 - \mu(\Omega)$ and $p = 2 + \mu(\Omega)$.

Proof. The proof is similar to that of Theorem 5. □

Example 5. Let $\Omega = [0, 3]$ and consider the cycle graph C_3. Then $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ is a grey graph on the graph C_3, depicted in Figure 8. This example shows that, if one part of the conditions of Theorem 7 is not satisfied, then the property of being one-valued cannot be deduced.

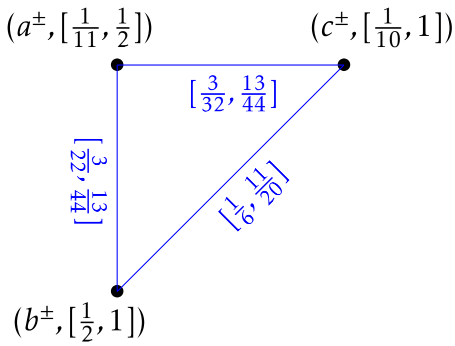

Figure 8: The grey graph $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ on C_3.

Theorem 8. Let Ω be a universe. Consider a graph $G^{\pm} = (V, E)$. Also, let $xy \in E$ and $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ be a grey graph. Then, $\mu^{\pm}(xy)$ is zero and one-valued if and only if one of the following statements is true.

(i) $\{(x^{\pm}, [0, 0]), (y^{\pm}, [\alpha, \beta]) : \alpha, \beta \in \mathbb{R}\} \subseteq \sigma^{\pm}$.

(ii) $\{(x^{\pm}, [\alpha, \alpha]), (y^{\pm}, [-\beta, \beta]) : \alpha, \beta \in \mathbb{R}\} \subseteq \sigma^{\pm}$.

(iii) $\{(x^{\pm}, [-\alpha, \alpha]), (y^{\pm}, [\beta, \beta]) : \alpha, \beta \in \mathbb{R}\} \subseteq \sigma^{\pm}$.

(iii) $\{(y^{\pm}, [0, 0]), (x^{\pm}, [\alpha, \beta] : \alpha, \beta \in \mathbb{R}\} \subseteq \sigma^{\pm}$.

Proof. Let $xy \in E$. Since $[\mu^-(xy), \mu^+(xy)] = 0$, $\mu^-(xy) = 0 = \mu^+(xy)$. Now, $\mu^-(xy) = 0$ implies $T_{min}(g^\circ(x^\pm), g^\circ(y^\pm)) = 0$. Also, since $\mu^-(xy) \geq 0$, $g^\circ(x^\pm) = 0$ or $g^\circ(y^\pm) = 0$. In addition, $\mu^+(xy) = 0$ implies $T_{min}(|Av(x^\pm)|, |Av(y^\pm)|) = 0$, and since $\mu^+(xy) \geq 0$, we get $|Av(x^\pm)| = 0$ or $|Av(y^\pm)| = 0$. Thus, $\big(g^\circ(x^\pm) = 0$ and $|Av(x^\pm)| = 0\big)$, $\big(g^\circ(x^\pm) = 0$ and $|Av(y^\pm)| = 0\big)$, $\big(g^\circ(y^\pm) = 0$ and $|Av(x^\pm)| = 0\big)$ or $\big(g^\circ(y^\pm) = 0$ and $|Av(y^\pm)| = 0\big)$. Hence, by solving the above equations we get $\big(x^+ = x^-$ and $x^+ = -x^-\big)$, $\big(x^+ = x^-$ and $y^+ = -y^-\big)$, $\big(y^+ = y^-$ and $x^+ = -x^-\big)$ or $\big(y^+ = y^-$ and $y^+ = -y^-\big)$. □

4 Discrete Grey Vertices in Grey Graphs

In this section, we prove some results on discrete grey vertices in grey graphs.

Definition 6. Let Ω be a universe and $x^\pm \subseteq \Omega$. Then, $x^\pm$ is called 2-*polar*, if $T_{pr}(x^-, x^+) < 0$, as shown in Figure 9.

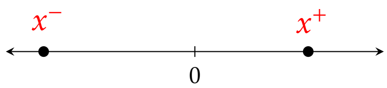

Figure 9: Position of the grey number $x^\pm$ on the real line (2-polar).

Proposition 1. *The following statements are true.*

(*i*) *If* $\mu(\Omega) = 2$ *and,* $x^\pm$ *and* $y^\pm$ *are 2-polar, then* $(xy)^\pm \notin \mu^\pm$.

(*ii*) *If* $\mu(\Omega) < 2$, $x^+ > (\dfrac{2+\mu(\Omega)}{2-\mu(\Omega)})x^-$ *and* $y^+ > (\dfrac{2+\mu(\Omega)}{2-\mu(\Omega)})y^-$, *then* $(xy)^\pm \notin \mu^\pm$.

(*iii*) *If* $Av(x^\pm) = 0$, *then for all* $xy \in E$, $(xy)^\pm \notin \mu^\pm$.

(*iv*) *If* $Av(x^\pm) \neq 0$ *and* $g^\circ(x^\pm) \leq Av(x^\pm)$, *then for all* $xy \in E$, $(xy)^\pm \in \mu^\pm$.

(*v*) *If* $x^\pm$ *is a non-zero and one-valued vertex, and* $y^\pm$ *is a grey vertex, then for all* $xy \in E$,
$(xy)^\pm \in [0, T_{min}(|x|, |Av(y^\pm)|]$.

(*vi*) *If* $y^\pm$ *is a non-zero and one-valued vertex, and* $x^\pm$ *is a grey vertex, then for all* $xy \in E$,
$(xy)^\pm \in [0, T_{min}(|y|, |Av(x^\pm)|]$.

(*vii*) *If* $x^\pm, y^\pm$ *are non-zero and one-valued vertices, then for all* $xy \in E$, $(xy)^\pm \in [0, T_{min}(|x|, |y|)]$.

Proof. (*i*) Let $xy \in E$. Since $x^\pm$ and $y^\pm$ are 2-polar,

$$x^- - x^+ < x^- + x^+ < x^+ - x^- \text{ and } y^- - y^+ < y^- + y^+ < y^+ - y^-.$$

Then $|x^- + x^+| < x^+ - x^-, |y^- + y^+| < y^+ - y^-$ and using $\mu(\Omega) = 2$, we get

$$g^\circ(x^\pm) > |Av(x^\pm)|$$

and

$$g^\circ(y^\pm) > |Av(y^\pm)|.$$

Thus, $T_{min}(g^\circ(x^\pm), g^\circ(y^\pm)) > T_{min}(|Av(x^\pm)|, |Av(y^\pm)|)$, and so $\mu^-(xy) > \mu^+(xy)$, which means $(xy)^\pm \notin \mu^\pm$.

(ii) Let $xy \in E$. Since $x^+ > (\frac{2+\mu(\Omega)}{2-\mu(\Omega)})x^-$ and $y^+ > (\frac{2+\mu(\Omega)}{2-\mu(\Omega)})y^-$, using $\mu(\Omega) < 2$ we get

$$\frac{2(x^- - x^+)}{\mu(\Omega)} < x^- + x^+ < \frac{2(x^+ - x^-)}{\mu(\Omega)} \text{ and } \frac{2(y^- - y^+)}{\mu(\Omega)} < y^- + y^+ < \frac{2(y^+ - y^-)}{\mu(\Omega)}.$$

Then,

$$\frac{|x^- + x^+|}{2} < \frac{x^+ - x^-}{\mu(\Omega)}, \frac{|y^- + y^+|}{2} < \frac{y^+ - y^-}{\mu(\Omega)}.$$

So, $g^\circ(x^\pm) > |Av(x^\pm)|$ and $g^\circ(y^\pm) > |Av(y^\pm)|$. Therefore,

$$T_{min}(g^\circ(x^\pm), g^\circ(y^\pm)) > T_{min}(|Av(x^\pm)|, |Av(y^\pm)|).$$

Thus, $\mu^-(xy) > \mu^+(xy)$, which means $(xy)^\pm \notin \mu^\pm$.

(iii) Let $xy \in E$. If $x^\pm = 0$ is a one-valued number, then $[\mu^-, \mu^+] = 0$ and so, $(xy)^\pm \notin \mu^\pm$. Let $x^\pm \neq 0$. Then $Av(x^\pm) = 0$ implies $x^+ = -x^- \neq 0$. Using $g^\circ(x^\pm) > 0$, we conclude that $[\mu^-, \mu^+] = \emptyset$. So, $(xy)^\pm \notin \mu^\pm$.

(iv) Let $xy \in E$. Since $g^\circ(x^\pm) \leq Av(x^\pm)$, we get

$$T_{min}(g^\circ(x^\pm), g^\circ(y^\pm)) \leq T_{min}(|Av(x^\pm)|, |Av(y^\pm)|).$$

Thus, $(xy)^\pm \in \mu^\pm$.

(v) Let $xy \in E$. Since $x^\pm$ is a non-zero and one-valued vertex, and $y^\pm$ is a grey vertex, we get $g^\circ(x^\pm) = 0$ and $Av(x^\pm) = x$. It follows that $(xy)^\pm \in [0, T_{min}(|x|, |Av(y^\pm)|]$.

(vii) This immediately follows from (v) and (vi). □

Theorem 9. Let Ω be a universe, $\mu(\Omega) \leq 1$, consider a graph $G^\pm = (V, E)$, and let $xy \in E$. If $x^\pm$ and $y^\pm$ are 2-polar, then $(xy)^\pm \notin \mu^\pm$.

Proof. Let $xy \in E$. Since $x^- < 0 < x^+$, $|x^+ - x^-| > |x^- + x^-|$. Using $\mu(\Omega) \leq 1$, we get

$$g^\circ(x^\pm) = \frac{x^+ - x^-}{\mu(\Omega)} \geq (x^+ - x^-) > |x^- + x^-| > |\frac{x^- + x^-}{2}|.$$

Then, $g^\circ(x^\pm) > |Av(x^\pm)|$.
Similarly, $g^\circ(y^\pm) > |Av(y^\pm)|$, since $y^\pm$ is 2-polar. Thus, $\mu^-(xy) > \mu^+(xy)$ and so, $(xy)^\pm \notin \mu^\pm$. □

Example 6. Let $G^\pm = (V, E)$ be the path graph P_2, $\Omega = [-0.5, 0.5]$, $x^\pm \in [-0.3, 0.3]$ and $y^\pm \in [-0.5, 0.4]$. Then $(xy)^\pm \notin \mu^\pm$.

In the following theorem, we describe a system $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ that cannot be a grey graph, and a fuzzy graph at the same time, because grey graphs and fuzzy graphs are based on non-null graphs.

Theorem 10. Let Ω be a universe, $G^{\pm} = (V, E)$ be a graph, and $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ be a grey graph on $G^{\pm}$. Then, $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ cannot be a fuzzy graph.

Proof. Let $xy \in E$ be an arbitrary edge. If $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ is a fuzzy graph on $G^{\pm}$, then there are $a, b, c \in \Omega$ such that $x^{\pm} = [a, a], y^{\pm} = [b, b]$ and $(xy)^{\pm} = [c, c]$, where $c \leq T_{min}(a, b)$. In this case, since $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ is a grey number-based graph on $G^{\pm}$, we get $\mu^{\pm} = 0$. Applying Proposition 1(iii), for any $x, y \in V$ we conclude $(xy)^{\pm} = 0$. So, there is $n \in \mathbb{N}$ such that $G \cong N_n$. □

5 Some Applications of Grey Graphs

The goal of this section is to present some applications of grey graphs in the real world.

Complex Computer Networks

Let $V = \{Pc1, Pc2, Pc3, Pc4\}$ be a set of computer systems, which have performance errors as in Table 1 (which are unknown to us), and changes based on various factors, including the passage of time and energy fluctuations, which we denote by a grey number $[a, b] \subseteq [0, 100] = \Omega$. For instance, $(Pc2)^{\pm} \in [10, 30]$ means that the performance error of system Pc2 changes from 10% to 30%. We want to construct a complex network via these computer systems and check their performance in the link of this network via various factors, including the passage of time, energy fluctuations, and the relative measure of the processing power of systems, as shown in Figure 10. In conclusion, with the link of systems Pc4 and Pc1, we get $(Pc1Pc4)^{\pm} \in [\frac{3}{10}, 15]$, which means that the performance error is reduced from 1% to 0.3%. On the other hand, system Pc2 is a weak system, because its performance error changes from 25% to 75%. Thus, as shown in Figure 10, this complex network is optimized by linking the computer systems.

Table 1: A complex computer network based on grey numbers

Computer System	$Pc1$	$Pc2$	$Pc3$	$Pc4$
Performance Error	$[0, 30]$	$[10, 30]$	$[25, 50]$	$[25, 75]$

Complex Economic Networks

Let $V = \{M1, M2, M3, M4\}$ be a set of industrial factories in an industrial area, which have (Dollar-based) economic outcomes in each hour as shown in Table 2, and change

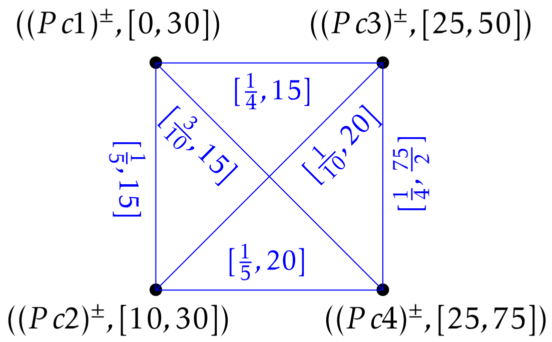

Figure 10: The grey graph $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ on C_4.

due to the fluctuations of the production and consumption market, energy price fluctuations, and fluctuations in the quality of raw materials, which we denote by a grey number $[a, b] \subseteq [-0.4, 1.5] = \Omega$. For instance, $(M1)^{\pm} \in [-0.2, 1.2]$ means that the industrial factory M1 can lose 0.2 Dollars and benefit 1.2 Dollars at most each hour. Also, $(M3)^{\pm} \in [0.4, 0.6]$ means that the industrial factory M3 can benefit from 0.4 Dollars to 0.6 Dollars in each hour. We want to optimize the resulting losses of an industrial area for the link of these industrial factories in a complex network, and the relative measure of the economic outcome of these factories in each hour, as depicted in Figure 11.

Table 2: A complex economic network based on grey numbers

Industrial Factory	$M1$	$M2$	$M3$	$M4$
Economic Outcome	[-0.2, 1.2]	$[-0.1, 0.4]$	$[0.4, 0.6]$	$[0.5, 0.7]$

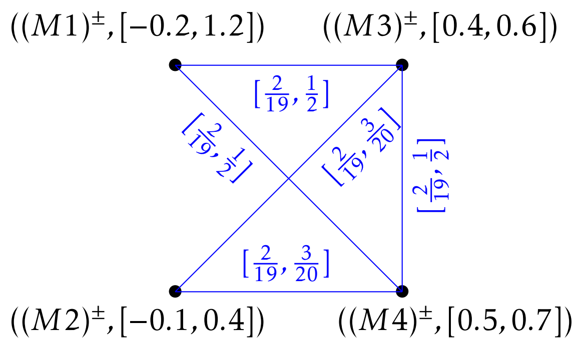

Figure 11: The grey graph $G = (\Omega, \sigma^{\pm}, \mu^{\pm})$ on C_4.

As can be seen from Figure 11, the industrial factories M1 and M2 cannot be linked, and $(M1M3)^{\pm} \in [\frac{2}{19}, \frac{1}{2}]$ means that the industrial factories M1 and M3 are profitable in this complex network, while out of the network, M1 and M3 can lose at most 0.2 Dollars and 0.1 Dollars in each hour, respectively.

Critical Complex Networks in Nature

In practice, by a grey number, we mean an indeterminate number whose possible value can be taken within a broad set of numbers or an interval. The theory of grey systems provides effective models for systems with both known and unknown data. Complex networks such as weighted graphs play a fundamental role in some problems in the real world, including social, economic, cultural, and natural issues. Primarily, grey graphs cover the weaknesses of ordinary networks, complex networks, and weighted networks. Hence, we can apply grey graphs in the design of different networks based on graphs and fuzzy graphs, which have weaknesses in facing critical situations in nature.

Suppose that we want to check a transportation network between two cities, A and B, a path, A, C, D, E, B of cities. Usually, the distance-time between these cities is n hours. However, with a natural disaster, the distance-time between these two cities is more than n hours. So, we cannot use a fuzzy graph in this way, because with changes in the conditions, the transportation graph cannot be changed. However, we can design the same network based on a grey graph.

6 Conclusion and Ideas for Further Work

This research analyzed and optimized some real problems on the novel notion of grey graph based on a given graph. Indeed, we proposed a new extension of graphs supported by grey numbers. Also, we found some properties by solving linear inequalities based on the averages and relative measures of grey numbers. We observed the difference between the concepts of a fuzzy graph and grey graph, which was based on a fundamental difference between grey numbers and fuzzy subsets. Moreover, we proved that the weaknesses of fuzzy graphs could be covered by grey graphs. Finally, we presented some optimization techniques and their application and some usages of grey graphs in the real world.

In future studies, we try to establish different results on hypergraphs based on grey numbers, as a generalization of graphs based on grey numbers. Also, we try to obtain some results on complete grey graphs, traceable grey graphs, Hamiltonian grey graphs, Eulerian grey graphs, and their usages.

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