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Research Article

Fixed Point Results under (ϕ, ψ)-Contractive Conditions and their Application in Optimization Problems

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Abstract. Fixed point theorems can be used to prove the solvability of optimization problems, differential equations and equilibrium problems, and the intrinsic flexibility of probabilistic metric spaces makes it possible to extend the idea of contraction mapping in several inequivalent ways. In this paper, we extend very recent fixed point theorems in the setting of Menger probabilistic metric spaces. We present some fixed point theorems for self-mappings satisfying a generalized (ϕ, ψ)-contractive condition in Menger probabilistic metric spaces which are contractions used extensively in global optimization problems. On the other hand, we consider a more general class of auxiliary functions in the contractivity condition and prove the existence of fixed points of non-expansive mappings on Menger probabilistic metric spaces.

Keywords. Compatible mappings, Fixed point, Optimization, Menger probabilistic metric spaces, Reciprocal continuity, Weak reciprocal continuity.

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1 Introduction

The study of metric fixed point theory has been researched extensively in the past decades, since fixed point theory plays a major role in mathematics and applied sciences, such as optimization, mathematical models, and economic theories. In 1922, a Polish mathematician, Banach, proved a famous result called Banach contraction principle in the context of fixed point theory [3]. Later, most of the authors intensively introduced many works regarding the fixed point theory in various spaces that have been widely used in different engineering applications including computational electromagnetism, problems of functional analysis e.g., iterative solutions of sets of linear equations, Fredholm and Volterra integral equations, ordinary differential equations. See [5, 6].

Fixed point theorems are developed for single-valued or set-valued mappings of metric spaces, topological vector spaces, posets and lattices, Banach lattices, etc. Among the themes of fixed point theory, the topic of approximation of fixed points of mappings is particularly important because it is useful for proving the existence of fixed points of mappings. It can be applied to prove the solvability of optimization problems, differential equations, variational inequalities, and equilibrium problems.

It is well-known that the probabilistic version of the classical Banach contraction principle was proved in 1972 by Sehgal and Bharucha-Reid [23]. In 2010, a probabilistic version of the Banach fixed point principle for general nonlinear contractions was established by Jacek Jachymski [11]. Also, the fixed point theorems in probabilistic metric spaces for other contraction mappings were investigated by many authors. See [9, 10, 22].

The concept of a Menger probabilistic metric space was first defined by Menger [15]. The idea of Menger was to use a distribution function instead of a nonnegative number for the value of a metric. M. De la Sen and E. Karapınar [7] discussed the properties of convergence of distances of p-cyclic contractions on the union of the p subsets of an abstract set X defining probabilistic metric spaces and Menger probabilistic metric spaces as well as the characterization of Cauchy sequences which converge to the best proximity points. The existence and uniqueness of fixed points and best proximity points of p-cyclic contractions, defined in induced complete Menger probabilistic metric spaces, are also discussed in the case that the associate complete metric space is a uniformly convex Banach space. Finally, the fixed points of the p-composite mappings restricted to each of the p subsets in the cyclic framework disposal are investigated. Also, in recent times, many fixed point theorems have been presented in the setting of a probabilistic metric space $(X, F, \triangle)$ in which F is a distance distribution function. Most of the results were inspired by their corresponding results on metric spaces. One of the most attractive, effective ways to introduce contractivity conditions in the probabilistic framework is based on considering some terms like in the following expression (see [16, 26]).

$$\frac{1}{F_{x,y}(t)} - 1, \quad where \quad x, y \in X \; and \; t > 0.$$

In this paper, we consider more general contractivity conditions replacing the function $t \to \frac{1}{t} - 1$ by an appropriate function h to establish the existence of a fixed point and its uniqueness for a self mapping and a common fixed point, or a coincidence point of two self-mappings in Menger probabilistic metric spaces. Also, we establish the existence of a coupled coincidence point and a common coupled fixed point for two self mappings a satisfying generalized (ϕ, ψ)-contractive condition in Menger probabilistic metric spaces. Our results generalize Theorem 2.1 and Theorem 2.2 of [4] and the some corollaries of [2, 7, 8, 24].

Before proving our main results, we recall some basic definitions and facts which will be used later in this paper.

2 Notations and Preliminaries

Definition 1. [10] A function $f : (-\infty, \infty) \to [0, 1]$ is called a distribution function, if it is nondecreasing and left continuous with $\inf_{x \in \mathbb{R}} f(x) = 0$. If in addition $f(0) = 0$, then f is called a distance distribution function. Furthermore, a distance distribution function f satisfying $\lim_{t\to\infty} f(t) = 1$ is called a Menger distance distribution function.

The set of all Menger distance distribution functions is denoted by Λ^+.

Definition 2. [10] A triangular norm (abbreviated as a T-norm) is a binary operation $\triangle$ on $[0, 1]$, which satisfies the following conditions.

a. $\triangle$ is associative and commutative.

b. $\triangle$ is continuous.

c. $\triangle(a, 1) = a$ for all $a \in [0, 1]$.

d. $\triangle(a, b) \le \triangle(c, d)$ whenever $a \le c$ and $b \le d$, for each $a, b, c, d \in [0, 1]$.

Among the important examples of a T-norm we mention the following two T-norms:

$$\triangle_p(a, b) = ab \quad \text{and} \quad \triangle_m(a, b) = \min\{a, b\},$$

where the T-norm $\triangle_m$ is the strongest T-norm, that is, $\triangle \le \triangle_m$ for every T-norm $\triangle$.

Definition 3. [9] A triangular norm $\triangle$ is said to be of H-type ($Hadžić$ type) if a family of functions $\{\triangle^n(t)\}_{n=1}^{+\infty}$ is equicontinuous at $t = 1$, that is,

$$\forall \varepsilon \in (0, 1), \exists \delta \in (0, 1) : t > 1 - \delta \Rightarrow \triangle^n(t) > 1 - \varepsilon \quad (n \ge 1),$$

where $\triangle^n : [0, 1] \to [0, 1]$ is defined as follows:

$$\triangle^1(t) = \triangle(t, t), \quad \triangle^n(t) = \triangle(t, \triangle^{n-1}(t)), \quad n = 2, 3, \dots .$$

Obviously, $\triangle^n(t) \le t$ for any $n \in \mathbf{N}$ and $t \in [0, 1]$.

Definition 4. [22] A Menger probabilistic metric space (abbreviated as a Menger PM space) is a triple $(X, F, \triangle)$ where X is a nonempty set, $\triangle$ is a continuous T-norm and F is a mapping from $X \times X$ into Λ^+ such that, if $F_{p,q}$ denotes the value of F at the pair (p, q), the following conditions hold.

- (PM_1) $F_{p,q}(t) = 1$ for all $t > 0$ if and only if $p = q$ $(p, q \in X)$.
- (PM_2) $F_{p,q}(t) = F_{q,p}(t)$ for all $t > 0$ and $p, q \in X$.
- (PM_3) $F_{p,r}(s+t) \geq \triangle(F_{p,q}(s), F_{q,r}(t))$ for all $p, q, r \in X$ and every $s > 0, t > 0$.

Definition 5. [22] A sequence $\{x_n\}$ in a Menger PM space X is said to converge to a point x in X (written as $x_n \to x$), if for every $\epsilon > 0$ and $\lambda \in (0,1)$, there is an integer $N(\epsilon,\lambda) > 0$ such that $F_{x_n,x}(\epsilon) > 1-\lambda$ for all $n \geq N(\epsilon,\lambda)$. The sequence is said to be a Cauchy sequence if for each $\epsilon > 0$ and $\lambda \in (0,1)$, there is an integer $N(\epsilon,\lambda) > 0$ such that $F_{x_n,x_m}(\epsilon) > 1-\lambda$, for all $n, m \geq N(\epsilon,\lambda)$. A Menger PM space $(X, F, \triangle)$ is said to be complete if every Cauchy sequence in X converges to a point of X. Also, the sequence is said to be a G-Cauchy sequence if for each $\epsilon > 0$ and $\lambda \in (0,1)$, there is an integer $N(\epsilon,\lambda) > 0$ such that $F_{x_{n+p},x_n}(\epsilon) > 1-\lambda$, for all $n \geq N(\epsilon,\lambda)$ and $p \in \mathbb{N}$. A Menger PM space $(X, F, \triangle)$ is said to be G-complete if every G-Cauchy sequence in X converges to a point of X.

It is easy to see that, for $\tilde{a} = (x, y), \tilde{b} = (u, v) \in X^2 = X \times X$, the function $\tilde{F}$ from X^2 into Λ^+, is a distribution function:

$$\tilde{F}_{\tilde{a},\tilde{b}}(t) = \min\{F_{x,u}(t), F_{y,v}(t)\} \quad for\ all\ t > 0.$$

Lemma 1. [12] If $(X, F, \triangle)$ is a complete Menger PM space, then $(X^2, \tilde{F}, \triangle)$ is also a complete Menger PM space.

Definition 6. [1]

i. Let f and g be two maps from X into Y. We say f and g have a coincidence point, if there exists a point x in X such that $fx = gx$.

ii. Let f and g be two self maps on X. We say $x \in X$ is a common fixed point of f and g, if $fx = gx = x$.

iii. An element $(x, y) \in X \times X$ is called a coupled point of a mapping $T : X \times X \to X$, if $T(x, y) = x$ and $T(y, x) = y$.

iv. An element $(x, y) \in X \times X$ is called a coupled coincidence point of mappings $T : X \times X \to X$ and $g : X \to X$ if $T(x, y) = gx$ and $T(y, x) = gy$

v. An element $(x, y) \in X \times X$ is called a common coupled fixed point of mappings $T : X \times X \to X$ and $g : X \to X$ if $T(x, y) = gx = x$ and $T(y, x) = gy = y$.

Definition 7. [8] Let f and g be two self maps of a Menger PM space $(X, F, \triangle)$. Then f and g are said to be Menger compatible if $\lim_{n\to\infty} F_{fgx_n,gfx_n}(t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x \in X$.

In 1982, Sessa [25] introduced the notion of weak commutativity condition for a pair of single-valued maps. Later, Jungek [13] generalized the concept of weak commutativity by introducing the notion of compatibility of maps. Pant [18] introduced point wise R-weakly commutativity of maps for noncompatible maps.

Two self mappings f and g of a metric space (X,d) are called R-weakly commuting of type-(A_g) [20], if there exists some positive real number R such that $d(ffx,gfx) \leq Rd(fx,gx)$ for all $x \in X$. Similarly, two self mappings f and g of a metric space (X,d) are called R-weakly commuting of type-(A_f) [20], if there exists some positive real number R such that $d(fgx,ggx) \leq Rd(fx,gx)$ for all $x \in X$.

In 2007, Kohali and Vashistha [14] introduced the notion of R-weakly commuting mappings in probabilistic metric spaces as follows.

Definition 8. Two self mappings f and g of a Menger PM space $(X,F,\triangle)$ are called R-weakly commuting of type-(MA_g), if there exists some real number $R \geq 0$ such that $F_{ffx,gfx}(t) \geq F_{fx,gx}(\frac{t}{R})$ for all $t > 0$ and $x \in X$.

In 1998, Pant [19] introduced the concept of reciprocal continuity for a pair of single-valued maps. In what follows, we have the same definition, but in a Menger PM space X.

Definition 9. Two self mappings f and g of a Menger PM space X are called reciprocally continuous, if $\lim_{n\to\infty} gfx_n = gx$ and $\lim_{n\to\infty} fgx_n = fx$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some $x \in X$.

Note that two reciprocally continuous mappings need not be continuous even at their common fixed point (see e.g., [19]).

Pant et al. [20] generalized reciprocal continuity by introducing the notion of weak reciprocal continuity for a pair of single-valued maps as follows, but in a metric space (X,d).

Definition 10. [8] Two self-mappings f and g of a Menger PM space X are called weak reciprocally continuous, if $\lim_{n\to\infty} gfx_n = gx$ or $\lim_{n\to\infty} fgx_n = fx$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some $x \in X$.

It seems important to note that reciprocal continuity implies weak reciprocal continuity, but the converse is not true (see Example 7 [8]).

Definition 11. Let Φ be the family of all functions $\phi : [0,\infty) \to [0,\infty)$ satisfying:

1. $\phi(t) = 0$ if and only if $t = 0$;
2. $\lim_{t\to\infty} \phi(t) = \infty$;
3. ϕ is continuous at $t = 0$.

Definition 12. Let Ψ be the class of all functions $\psi : [0,\infty) \to [0,\infty)$ satisfying:

1. ψ is nondecreasing;
2. $\psi(0) = 0$;
3. if $\{a_n\} \subset [0,\infty)$ is a sequence such that $\{a_n\} \to 0$, then $\{\psi^n(a_n)\} \to 0$ (where ψ^n denotes the nth-iterate of ψ).

We recall that ψ is continuous at $t=0$ for functions in Ψ (Proposition 7 of [21]). The following family of auxiliary functions was introduced in [21].

Definition 13. Let $\mathcal{H}$ be the family of all functions $h:(0,1]\to[0,\infty)$ satisfying:

- $(\mathcal{H}_1)$ if $\{a_n\}\subset(0,1]$, then $\{a_n\}\to 1$ if and only if $\{h(a_n)\}\to 0$;
- $(\mathcal{H}_2)$ if $\{a_n\}\subset(0,1]$, then $\{a_n\}\to 0$ if and only if $\{h(a_n)\}\to\infty$.

The previous conditions are guaranteed when $h:(0,1]\to[0,\infty)$ is a strictly decreasing bijection between $(0,1]$ and $[0,\infty)$ such that h and h^{-1} are continuous (in a broad sense, it is sufficient to assume the continuity of h and h^{-1} on the extremes of the respective domains). For instance, this is the case for the function $h(t)=1/t-1$ for all $t\in(0,1]$. However, the functions in $\mathcal{H}$ need not be continuous, or monotone.

Proposition 1. [21] If $h\in\mathcal{H}$, then $h(1)=0$. Furthermore, $h(t)=0$ if and only if $t=1$.

3 The Main Results

In this section we extend the fixed point theorems in several ways: the metric space is more general, the contractivity condition is better and the involved auxiliary functions form a wider class.

Theorem 1. Let $(X,F,\triangle)$ be a Menger PM space with a T-norm $\triangle$ of H-type, T,S be two self-maps of X such that for some $c\in(0,1),\phi\in\Phi,\psi\in\Psi$, and $h\in\mathcal{H}$ satisfying

$$h(F_{Tx,Ty}(\phi(ct)))\leq\psi(h(M_S(x,y))),\tag{1}$$

for any $x,y\in X$ and all $t>0$ and

$$M_S(x,y)=max\{F_{Sx,Sy}(\phi(t)),\frac{\triangle(F_{Sx,Tx}(\phi(t)),F_{Sy,Ty}(\phi(t)))}{1+F_{Tx,Ty}(\phi(t))}\},\tag{2}$$

with $T(X)\subseteq S(X)$, then T and S have a coincidence point in X if either

a. X is G-complete and S is surjective; or,

b. X is G-complete and S is continuous and T and S are Menger compatible; or,

c. $S(X)$ is G-complete; or,

d. $T(X)$ is G-complete.

Furthermore, if $h\in\mathcal{H}$ is decreasing, the coincidence point is unique, i.e., $Sp=Sq$ whenever $Sp=Tp$ and $Sq=Tq$ $(p,q\in X)$.

Proof. Let $x_0 \in X$. Set $Tx_0 = y_1$. Since $T(X) \subseteq S(X)$, choose x_1 such that $y_1 = Sx_1 = Tx_0$. In general, choose x_{n+1} such that $y_{n+1} = Sx_{n+1} = Tx_n$.

From (1), we obtain

$$h(F_{y_n,y_{n+1}}(\phi(ct))) = h(F_{Tx_{n-1},Tx_n}(\phi(ct))) \leq \psi(h(M_S(x_{n-1},x_n))),$$

for all $n \in \mathbb{N}$ and all $t > 0$ and by (2)

$$\begin{aligned} M_S(x_{n-1},x_n) &= max\{F_{Sx_{n-1},Sx_n}(\phi(t)), \\ &\quad \frac{\triangle(F_{Sx_{n-1},Tx_{n-1}}(\phi(t)), F_{Sx_n,Tx_n}(\phi(t)))}{1+F_{Tx_{n-1},Tx_n}(\phi(t))}\} \\ &= max\{F_{y_{n-1},y_n}(\phi(t)), \frac{\triangle(F_{y_{n-1},y_n}(\phi(t)), F_{y_n,y_{n+1}}(\phi(t)))}{1+F_{y_n,y_{n+1}}(\phi(t))}\} \\ &= F_{y_{n-1},y_n}(\phi(t)). \end{aligned}$$

So, we obtain

$$h(F_{y_n,y_{n+1}}(\phi(ct))) \leq \psi(h(F_{y_{n-1},y_n}(\phi(t)))), \tag{3}$$

for all $n \in \mathbb{N}$ and all $t > 0$. We claim that $\lim_{n\to\infty} F_{y_n,y_{n+1}}(s) = 1$ for all $s > 0$.

To prove it, let $s > 0$ be arbitrary. As $\lim_{r\to\infty} c^r s = 0$ and ϕ is continuous at $t = 0$, $\lim_{r\to\infty} \phi(c^r s) = \phi(0) = 0$; since $s > 0$, there exists $r \in \mathbb{N}$ such that

$$\phi(c^r s) \leq s.$$

Let $n \in \mathbb{N}$ be such that $n > r$. Applying the condition (3), it follows that

$$h(F_{y_n,y_{n+1}}(\phi(c^r s))) \leq \psi(h(F_{y_{n-1},y_n}(\phi(c^{r-1}s)))). \tag{4}$$

Repeating this argument, we find that

$$h(F_{y_{n-1},y_n}(\phi(c^{r-1}s))) \leq \psi(h(F_{y_{n-2},y_{n-1}}(\phi(c^{r-2}s)))).$$

Since ψ is nondecreasing,

$$\psi(h(F_{y_{n-1},y_n}(\phi(c^{r-1}s)))) \leq \psi^2(h(F_{y_{n-2},y_{n-1}}(\phi(c^{r-2}s)))). \tag{5}$$

Combining inequalities (4) and (5), we deduce that

$$\begin{aligned} h(F_{y_n,y_{n+1}}(\phi(c^r s))) &\leq \psi(h(F_{y_{n-1},y_n}(\phi(c^{r-1}s)))) \\ &\leq \psi^2(h(F_{y_{n-2},y_{n-1}}(\phi(c^{r-2}s)))). \end{aligned}$$

By repeating this argument n times, we have

$$\begin{aligned} h(F_{y_n,y_{n+1}}(\phi(c^r s))) &\leq \psi^n(h(F_{y_0,y_1}(\phi(c^{r-n}s)))) \\ &\leq \psi^n(h(F_{y_0,y_1}(\phi(\frac{s}{c^{n-r}})))), \end{aligned} \tag{6}$$

for all $n > r$. As a consequence,

$$\begin{aligned}\lim_{n\to\infty}\frac{s}{c^{n-r}}=\infty &\Rightarrow \lim_{n\to\infty}\phi(\frac{s}{c^{n-r}})=\infty\\ &\Rightarrow \lim_{n\to\infty}F_{y_0,y_1}(\phi(\frac{s}{c^{n-r}}))=1\\ &\Rightarrow \lim_{n\to\infty}h(F_{y_0,y_1}(\phi(\frac{s}{c^{n-r}})))=0.\end{aligned}$$

As $\{a_n=h(F_{y_0,y_1}(\phi(\frac{s}{c^{n-r}})))\}\to 0$, we have $\{\psi^n(a_n)\}\to 0$. Since $h\in\mathcal{H}$, by (6), we deduce that

$$\lim_{n\to\infty}h(F_{y_n,y_{n+1}}(\phi(c^r s)))=0.$$

In particular, as $h\in\mathcal{H}$, condition $(\mathcal{H}_1)$ implies that

$$\lim_{n\to\infty}F_{y_n,y_{n+1}}(\phi(c^r s))=1.$$

Taking $\phi(c^r s)<s$ into account, we observe that

$$F_{y_n,y_{n+1}}(\phi(c^r s))\le F_{y_n,y_{n+1}}(s)\le 1.$$

Therefore,

$$\lim_{n\to\infty}F_{y_n,y_{n+1}}(s)=1,$$

which means that $\{y_n\}$ is a G-Cauchy sequence in X by Lemma 15 of [21].

Case(a): Let X be G-complete and S be surjective. Then, by the completeness of X, $\{y_n\}$ converges to a point p in X. So, $\lim_{n\to\infty}Sx_{n+1}=\lim_{n\to\infty}Tx_n=p$. Hence, there exists a point z in X such that $p=Sz$.

Now, we will prove that $Tz=Sz$. From (1), we have

$$h(F_{Tx_n,Tz}(\phi(ct)))\le\psi(h(M_S(x_n,z))),$$

for all $n\in\mathbb{N}$ and all $t>0$ and by (2)

$$M_S(x_n,z)=max\{F_{Sx_n,Sz}(\phi(t)),\frac{\triangle(F_{Sz,Tz}(\phi(t)),F_{Sx_n,Tx_n}(\phi(t)))}{1+F_{Tx_n,Tz}(\phi(t))}\}.$$

Letting $n\to\infty$ and from $\lim\limits_{n\to\infty}Tx_n=\lim\limits_{n\to\infty}Sx_{n+1}=Sz$, we get

$$\begin{aligned}\lim_{n\to\infty}M_S(x_n,z)=1 &\Rightarrow \lim_{n\to\infty}h(M_S(x_n,z))=0\\ &\Rightarrow \lim_{n\to\infty}\psi(h(M_S(x_n,z)))=0.\end{aligned}$$

Therefore, $\lim\limits_{n\to\infty}h(F_{Tx_n,Tz}(\phi(ct)))=0$, and so, $\lim\limits_{n\to\infty}F_{Tx_n,Tz}(\phi(ct))=1$ for all $t>0$, so we conclude that $Tz=Sz$. Hence, z is the coincidence point of T and S.

Case(b): Since X is G-complete, $\{y_n\}$ converges to a point p in X. Suppose S is continuous and S and T are Menger compatible. Since $\lim_{n\to\infty}y_n=p$, we have $\lim_{n\to\infty}Sy_n=Sp$. Note that since $\lim_{n\to\infty}Sx_n=\lim_{n\to\infty}Tx_n=p$ and S and T are Menger compatible, $\lim_{n\to\infty}F_{STx_n,TSx_n}(t)=1$.

From (1), we have

$$h(F_{Tp,TSx_n}(\varphi(ct))\le\psi(h(M_S(p,Sx_n)),$$

for all $t > 0$ and

$$M_S(p, Sx_n) = max\{F_{Sp,SSx_n}(\phi(t)), \frac{\triangle(F_{Sp,Tp}(\phi(t)), F_{Sx_n,Tx_n}(\phi(t)))}{1 + F_{Tp,TSx_n}(\phi(t))}\}.$$

Taking limit as $n \to \infty$, by $\lim_{n\to\infty} SSx_n = Sy_n = Sp$, we get

$$\lim_{n\to\infty} M_S(p, Sx_n) = 1,$$

and hence,

$$\lim_{n\to\infty} F_{Tp,TSx_n}(t) = 1,$$

so, by (PM_3) we have,

$$F_{Tp,Sp}(t) \geq \triangle(F_{Tp,TSx_n}(\frac{t}{2}), \triangle(F_{TSx_n,STx_n}(\frac{t}{4}), F_{STx_n,Sp}(\frac{t}{4}))),$$

taking limit as $n \to \infty$, implies that $Sp = Tp$.

Case(c): In this case since $\{Sx_n\}$ is a sequence in $S(X)$ and $S(X)$ is G-complete, $\lim_{n\to\infty} y_n = Sx_n = p$ for some $p \in S(X)$. Let $p = Sz$ for some $z \in S^{-1}p$ and then the proof is complete by case (a).

Case(d): In this case, $p \in T(X) \subseteq S(X)$ and the proof is complete by case (a).

Uniqueness: Let q be another coincidence point of S and T, then by (1),

$$h(F_{Tp,Tq}(\phi(ct))) \leq \psi(h(M_S(p, q))), \tag{7}$$

for all $t > 0$ and

$$\begin{aligned} M_S(p, q) &= max\{F_{Sp,Sq}(\phi(t)), \frac{\triangle(F_{Sp,Tp}(\phi(t)), F_{Sq,Tq}(\phi(t))}{1 + F_{Tp,Tq}(\phi(t))}\} \\ &= max\{F_{Tp,Tq}(\phi(t)), \frac{1}{1 + F_{Tp,Tq}(\phi(t))}\}, \end{aligned} \tag{8}$$

and so, since h is decreasing, from (7) and (8), we get

$$h(F_{Tp,Tq}(\phi(ct))) \leq \psi(h(F_{Tp,Tq}(\phi(t)))),$$

for all $t > 0$. Also, we can write

$$h(F_{Tp,Tq}(\phi(t))) \leq \psi(h(F_{Tp,Tq}(\phi(\frac{t}{c})))).$$

Since ψ is nondecreasing, by repeating this argument, we deduce that

$$\begin{aligned} h(F_{Tp,Tq}(\phi(t))) &\leq \psi(h(F_{Tp,Tq}(\phi(\frac{t}{c})))) \\ &\leq \psi^2(h(F_{Tp,Tq}(\phi(\frac{t}{c^2})))) \\ &\vdots \\ &\leq \psi^n(h(F_{Tp,Tq}(\phi(\frac{t}{c^n})))), \end{aligned} \tag{9}$$

for all $t > 0$. On the other hand, from the hypotheses, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{t}{c^n}=\infty &\Rightarrow \lim_{n\to\infty}\phi(\frac{t}{c^n})=\infty\\ &\Rightarrow \lim_{n\to\infty}F_{Tp,Tq}(\phi(\frac{t}{c^n}))=1\\ &\Rightarrow \lim_{n\to\infty}h(F_{Tp,Tq}(\phi(\frac{t}{c^n})))=0\\ &\Rightarrow \lim_{n\to\infty}\psi(h(F_{Tp,Tq}(\phi(\frac{t}{c^n}))))=0,\end{aligned}$$

for all $t > 0$. So, from (9),

$$\lim_{n\to\infty}h(F_{Tp,Tq}(\phi(t)))=0,$$

and hence,

$$\lim_{n\to\infty}F_{Tp,Tq}(\phi(t))=1,$$

for all $t > 0$, and therefore, it is easy to conclude that $\lim_{n\to\infty}F_{Tp,Tq}(t)=1$, for all $t > 0$, which means that $Tp = Tq$ and $Sp = Sq$ by virtue of (PM_1). □

Corollary 1. Let $(X,F,\triangle)$ be a G-complete Menger PM space with a T-norm $\triangle$ of H-type, and T be a self-mapping of X satisfying (1) for some $c\in(0,1),\phi\in\Phi,\psi\in\Psi$, and $h\in\mathcal{H}$ with $S = I$, the identity map on X. Then T has a fixed point and is continuous at this fixed point.

Proof. The existence of the fixed point comes from Theorem 1 by setting $S = I$. To prove the continuity, let $\{y_n\}\subset X$ with $\lim_{n\to\infty}y_n = p$, where p is the fixed point of T.

Using (1), we have

$$h(F_{Tp,Ty_n}(\varphi(ct))\leq\psi(h(M(p,y_n))),$$

for all $t > 0$ and

$$M(p,y_n)=max\{F_{p,y_n}(\phi(t)),\frac{\triangle(F_{p,Tp}(\phi(t)),F_{y_n,Ty_n}(\phi(t))}{1+F_{Tp,Ty_n}(\phi(t))}\},$$

by taking limit as $n\to\infty$, we get $\lim_{n\to\infty}M(p,y_n)=1$, and so, we obtain

$$\lim_{n\to\infty}Ty_n=p=Tp.$$

□

Example 1. Let $X=[0,\infty)$. Define $F: X\times X\to\Lambda^+$ by

$$F_{x,y}(t)=\begin{cases}\epsilon_{max\{x,y\}}(t), & \text{if} \quad x\neq y,\\ 1, & \text{if} \quad x=y,\end{cases}$$

for all $x,y\in X$ and for all $t > 0$, such that

$$\epsilon_a(t) = \begin{cases} 0, & \text{if} \quad 0 \le t \le a, \\ 1, & \text{if} \quad a < t \le \infty. \end{cases}$$

It is easy to see that $(X, F, \triangle_p)$ is a G-complete Menger PM space (see Example 12, [21]). Let $T : X \to X$ be the self-mapping defined by $Tx = \frac{x}{2}$ for all $x \in X$.

Now, consider self-mappings ϕ and ψ on $[0,\infty)$ defined by $\psi(t) = \phi(t) = t$, for all $t \in [0,\infty)$, and let $h : (0,1] \to [0,\infty)$ be an arbitrary strictly decreasing bijection between $(0,1]$ and $[0,\infty)$ such that h and h^{-1} are continuous (for instance, $h(t) = 1/t - 1$ for all $t \in (0,1]$, but any other function satisfying these properties yields the same result). In this context, the contractivity conditions (1) and (2) are equivalent to

$$\begin{aligned} &h(F_{Tx,Ty}(\phi(ct))) \le \psi(h(M(x,y))) \\ &\Leftrightarrow h(F_{Tx,Ty}(ct)) \le h(M(x,y)) \\ &\Leftrightarrow F_{Tx,Ty}(ct) \ge M(x,y) \ge F_{x,y}(t), \end{aligned}$$

for all $x, y \in X$, $t > 0$ and for some $c \in (0,1)$. Let $x \ne y$ and by setting $c = \frac{1}{2}$, we get

$$\begin{aligned} F_{Tx,Ty}(ct) &= F_{\frac{x}{2},\frac{y}{2}}(\frac{t}{2}) \\ &= \epsilon_{max\{\frac{x}{2},\frac{y}{2}\}}(\frac{t}{2}) \\ &= \begin{cases} 0, & \text{if} \quad 0 \le \frac{t}{2} \le max\{\frac{x}{2},\frac{y}{2}\} \\ 1, & \text{if} \quad max\{\frac{x}{2},\frac{y}{2}\} < \frac{t}{2} \end{cases} \\ &= \begin{cases} 0, & \text{if} \quad 0 \le t \le max\{x,y\} \\ 1, & \text{if} \quad max\{x,y\} < t \end{cases} \\ &= F_{x,y}(t). \end{aligned}$$

It is clear that if $x = y$, then the contractivity condition is satisfied. Also, all the assumptions considered in Theorem 1 or Corollary 1 are satisfied and hence, it guarantees that T has a unique fixed point (which is $x = 0$) and it is continuous at the fixed point.

Definition 14. [8] Let $(X, F, \triangle)$ be a Menger PM space and $T : X \times X \to X$ and $g : X \to X$. Then T and g are Menger compatible if

$$\lim_{n\to\infty} F_{gT(x_n,y_n),T(gx_n,gy_n)}(t) = 1,$$

for all $t > 0$ and

$$\lim_{n\to\infty} F_{gT(y_n,x_n),T(gy_n,gx_n)}(t) = 1,$$

for all $t > 0$, whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\lim_{n\to\infty} T(x_n, y_n) = \lim_{n\to\infty} gx_n = x,$$

and

$$\lim_{n\to\infty} T(y_n, x_n) = \lim_{n\to\infty} gy_n = y,$$

for all $x, y \in X$.

Corollary 2. Let $(X,F,\triangle)$ be a Menger PM space with a T-norm $\triangle$ of H-type, $G: X\times X\to X$ and $f: X\to X$ be two mappings such that for some $c\in(0,1),\phi\in\Phi,\psi\in\Psi$, and $h\in\mathcal{H}$

$$h(F_{G(x,y),G(u,v)}(\phi(ct)))\leq\psi(h(M_f^*((x,y),(u,v)))), \tag{10}$$

for all $(x,y),(u,v)\in X\times X$ and all $t>0$ and

$$M_f^*((x,y),(u,v)) = max\{min\{F_{fx,fu}(\phi(t)),F_{fy,fv}(\phi(t))\}, \frac{\triangle(min\{F_{fx,G(x,y)}(\phi(t)),F_{fy,G(y,x)}(\phi(t))\},min\{F_{fu,G(u,v)}(\phi(t)),F_{fv,G(v,u)}(\phi(t))\})}{1+min\{F_{G(x,y),G(u,v)}(\phi(t)),F_{G(y,x),G(v,u)}(\phi(t))\}}\}, \tag{11}$$

with $G(X\times X)\subseteq f(X)$. Then G and f have a coupled coincidence point if either one of the conditions (a) or (b) or (c) in Theorem 1 holds, or $G(X\times X)$ is G-complete. Furthermore, if $h\in\mathcal{H}$ is decreasing, the coupled coincidence value is unique.

Proof. Let $\tilde{X}=X\times X$. It follows from Lemma 1 that $(\tilde{X},\tilde{F},\triangle)$ is also a Menger PM space, where

$$\tilde{F}_{\tilde{a},\tilde{b}}(t):=\min\{F_{x,u}(t),F_{y,v}(t)\},$$

for $\tilde{a}=(x,y),\tilde{b}=(u,v)\in\tilde{X}$.

The self-mappings T and $S:\tilde{X}\to\tilde{X}$ are defined as follows

$$T\tilde{a}=(G(x,y),G(y,x))\ for\ all\ \tilde{a}=(x,y)\in\tilde{X},$$

and

$$S\tilde{a}=(fx,fy)\ for\ all\ \tilde{a}=(x,y)\in\tilde{X}.$$

Then a coupled coincidence point of G and f is a coincidence point of T and S in $X\times X$ and vice versa. On the other hand, for all $t>0$ and $\tilde{a}=(x,y),\tilde{b}=(u,v)\in\tilde{X}$, from (11), we have

$$\begin{aligned}M_f^*((x,y),(u,v)) &= max\{\tilde{F}_{S\tilde{a},S\tilde{b}}(\phi(t)),\frac{\triangle(\tilde{F}_{S\tilde{a},T\tilde{a}}(\phi(t)),\tilde{F}_{S\tilde{b},T\tilde{b}}(\phi(t)))}{1+\tilde{F}_{T\tilde{a},T\tilde{b}}(\phi(t))}\}\\ &= M_S(\tilde{a},\tilde{b}),\end{aligned}$$

so, by (10) we have

$$h(F_{G(x,y),G(u,v)}(\phi(ct)))\leq\psi(h(M_S(\tilde{a},\tilde{b}))).$$

Similarly,

$$h(F_{G(y,x),G(v,u)}(\phi(ct)))\leq\psi(h(M_S(\tilde{a},\tilde{b}))).$$

Thus

$$h(\tilde{F}_{T\tilde{a},T\tilde{b}}(\phi(ct)))\leq\psi(h(M_S(\tilde{a},\tilde{b}))).$$

If X is G-complete, it follows from Lemma 1 that $(\tilde{X},\tilde{F},\triangle)$ is also a G-complete Menger PM space. Also, it is easy to see that all the conditions of Theorem 1 hold for two self mappings T and S on $X\times X$. Thus, following Theorem 1, we see that G and f have a coupled coincidence point, that is, there exist $p,q\in X$ such that $G(p,q)=fp$ and $G(q,p)=fq$. □

Following similar arguments as in the proof of Corollary 1 and Corollary 2, we deduce the next result. We omit the details of the proof.

Corollary 3. Let $(X,F,\triangle)$ be a G-complete Menger PM space with a T-norm $\triangle$ of H-type, $G: X\times X \to X$ be a mapping such that for some $c\in(0,1), \phi\in\Phi, \psi\in\Psi$, and $h\in\mathcal{H}$,

$$h(F_{G(x,y),G(u,v)}(\phi(ct)))\leq \psi(h(M^*((x,y),(u,v)))),$$

for all $(x,y),(u,v)\in X\times X$ and all $t>0$ and

$$M^*((x,y),(u,v)) = max\{min\{F_{x,u}(\phi(t)),F_{y,v}(\phi(t))\},\\ \frac{\triangle(min\{F_{x,G(x,y)}(\phi(t)),F_{y,G(y,x)}(\phi(t))\}, min\{F_{u,G(u,v)}(\phi(t)),F_{v,G(v,u)}(\phi(t))\})}{1+min\{F_{G(x,y),G(u,v)}(\phi(t)),F_{G(y,x),G(v,u)}(\phi(t))\}}.$$

Then G has a coupled point and it is continuous at this coupled point.

Theorem 2. Let $(X,F,\triangle)$ be a G-complete Menger PM space with a T-norm $\triangle$ of H-type, T and S be two weakly reciprocally continuous self maps of X satisfying (1) and (2) for some $c\in(0,1), \phi\in\Phi, \psi\in\Psi$, and $h\in\mathcal{H}$, with $T(X)\subseteq S(X)$, then T and S have a coincidence point in X (if h is decreasing, T and S have a common fixed point in X), if either

a. T and S are Menger compatible; or,

b. T and S are R-weakly commuting of type-(MA_S); or,

c. T and S are R-weakly commuting of type-(MA_T).

Proof. Following a similar argument as in the proof of Theorem 1, we deduce that $\{Sx_n\}$ and $\{Tx_n\}$ are G-Cauchy sequences in X and the G-completeness of the space implies $\lim_{n\to\infty} Sx_{n+1} = \lim_{n\to\infty} Tx_n = p$ for some $p\in X$. Since S and T are weakly reciprocally continuous, either $\lim_{n\to\infty} STx_n = Sp$ or $\lim_{n\to\infty} TSx_n = Tp$.

We prove the result in three cases.

Case(a): Let $\lim_{n\to\infty} STx_n = Sp$. Now using the Menger compatibility of S and T, we get $\lim_{n\to\infty} F_{STx_n,TSx_n}(t) = 1$ for all $t>0$. For $t>0$, we have $1\geq F_{TSx_n,Sp}(t)\geq \triangle(F_{TSx_n,STx_n}(\frac{t}{2}), F_{STx_n,sp}(\frac{t}{2}))$. Letting $n\to\infty$, we get $\lim_{n\to\infty} TSx_n = Sp$ and so $\lim_{n\to\infty} TSx_{n+1} = \lim_{n\to\infty} TTx_n = Sp$.

Now using (1),

$$h(F_{Tp,TTx_n}(\phi(ct)))\leq \psi(h(M_S(p,Tx_n))),$$

for all $t>0$ and

$$M_S(p,Tx_n) = max\{F_{Sp,STx_n}(\phi(t)), \frac{\triangle(F_{Sp,Tp}(\phi(t)), F_{STx_n,TTx_n}(\phi(t)))}{1+F_{Tp,TTx_n}(\phi(t))}\}.$$

On letting $n\to\infty$, we get $\lim_{n\to\infty} M_S(p,Tx_n) = 1$, and so, similar to the previous argument, we obtain

$$\lim_{n\to\infty} F_{Tp,TTx_n}(t) = 1,$$

for all $t>0$, hence, $\lim_{n\to\infty} TTx_n = Tp$. This implies that $Sp = Tp$. So p is the coincidence point of T and S.

Now, let h be decreasing. Since $\lim_{n\to\infty} TTx_n = \lim_{n\to\infty} STx_n = Sp$, by the same argument, the Menger compatibility of S and T implies commutativity at the coincidence point, hence $STp = TSp = SSp = TTp$.

Again, using (1)

$$h(F_{Tp,TTp}(\phi(ct))) \le \psi(h(M_S(p,Tp))),$$

for all $t>0$ and

$$M_S(p,Tp) = max\{F_{Sp,STp}(\phi(t)), \frac{\triangle(F_{Sp,Tp}(\phi(t)), F_{STp,TTp}(\phi(t)))}{1+F_{Tp,TTp}(\phi(t))}\},$$

since, h is decreasing and ψ is nondecreasing, we have

$$h(F_{Tp,TTp}(\phi(ct))) \le \psi(h(F_{Tp,TTp}(\phi(t)))),$$

so, by the same argument as in the proof of Theorem 1, we get $Tp = TTp$. Hence, $Tp = TTp = STp$, i.e., Tp is the common fixed point of S and T.

Next, suppose that $\lim_{n\to\infty} TSx_n = Tp$. Since $T(X) \subseteq S(X)$, $Tp = Sz$ for some $z \in X$ and $\lim_{n\to\infty} TSx_n = Sz$. The Menger compatibility of S and T implies $\lim_{n\to\infty} STx_n \to Sz$. Since $TSx_{n+1} = TTx_n$ and $TSx_{n+1} \to Sz$, it follows that $TTx_n \to Sz$.

Now using (1),

$$h(F_{Tz,TTx_n}(\phi(ct))) \le \psi(h(M_S(z,Tx_n))),$$

for all $t>0$ and

$$M_S(z,Tx_n) = max\{F_{Sz,STx_n}(\phi(t)), \frac{\triangle(F_{Sz,Tz}(\phi(t)), F_{STx_n,TTx_n}(\phi(t)))}{1+F_{Tz,TTx_n}(\phi(t))}\}.$$

On letting $n\to\infty$, we get $\lim_{n\to\infty} M_S(z,Tx_n) = 1$, and so, similar to the previous argument, we obtain

$$\lim_{n\to\infty} F_{Tz,TTx_n}(t) = 1,$$

for all $t>0$, hence, $\lim_{n\to\infty} TTx_n = Tz$. This implies that $Sz = Tz$. So z is the coincidence point of T and S.
(Now, let h be decreasing. By the same argument, the Menger compatibility of S and T implies commutativity at the coincidence point, hence $STz = TSz = TTz = SSz$. So, using a similar argument as the one above we obtain $Tz = TTZ = STz$. That is, Tz is the common fixed point of S and T.)

Case(b): Now suppose that S and T are R-weakly commuting of type-(MA_S). Since S and T are weakly reciprocally continuous, either $\lim_{n\to\infty} STx_n = Sp$ or $\lim_{n\to\infty} TSx_n = Tp$.

Let $\lim_{n\to\infty} STx_n = Sp$. The R-weak commutativity of type-(MA_S) of S and T gives $F_{STx_n,TTx_n}(t) \ge F_{Sx_n,Tx_n}(\frac{t}{R})$ for all $t>0$. Letting $n\to\infty$, we get $TTx_n = Sp$.

Now using (1), we get

$$h(F_{Tp,TTx_n}(\phi(ct))) \le \psi(h(M_S(p,Tx_n))),$$

for all $t > 0$ and

$$M_S(p, Tx_n) = max\{F_{Sp,STx_n}(\phi(t)), \frac{\triangle(F_{Sp,Tp}(\phi(t)), F_{STx_n,TTx_n}(\phi(t)))}{1 + F_{Tp,TTx_n}(\phi(t))}\}.$$

On letting $n \to \infty$, we get $\lim_{n\to\infty} M_S(p, Tx_n) = 1$, and so, similar to the previous argument, we obtain

$$\lim_{n\to\infty} F_{Tp,TTx_n}(t) = 1,$$

for all $t > 0$, hence, $\lim_{n\to\infty} TTx_n = Tp$. This implies that $Sp = Tp$. So, p is the coincidence point of T and S.

(Now, let h be decreasing. Again by the R-weak commutativity of type-(MA_S), $F_{TTp,STp}(t) \geq F_{Sp,Tp}(\frac{t}{R})$. This gives $TTp = STp = TSp = SSp$.

Using (1), we get

$$h(F_{Tp,TTp}(\phi(ct))) \leq \psi(h(M_S(p, Tp))),$$

for all $t > 0$ and

$$M_S(p, Tp) = max\{F_{Sp,STp}(\phi(t)), \frac{\triangle(F_{Sp,Tp}(\phi(t)), F_{STp,TTp}(\phi(t)))}{1 + F_{Tp,TTp}(\phi(t))}\},$$

since, h is decreasing and ψ is nondecreasing, we have

$$h(F_{Tp,TTp}(\phi(ct))) \leq \psi(h(F_{Tp,TTp}(\phi(t)))),$$

so, by the same argument as in the proof of Theorem (1), we get $Tp = TTp$. Hence, $Tp = TTp = STp$, i.e., Tp is the common fixed point of S and T.)

Now, suppose that $\lim_{n\to\infty} TSx_n = Tp$. Since $T(X) \subseteq S(X)$, $Tp = Sz$ for some $z \in X$ and $\lim_{n\to\infty} TSx_n = Sz$. Since $TSx_{n+1} = TTx_n$ and $TSx_{n+1} \to Sz$, it follows that $TTx_n \to Sz$. Then the R-weak commutativity of type-(MA_S) of S and T gives $F_{STx_n,TTx_n}(t) \geq F_{Sx_n,Tx_n}(\frac{t}{R})$ for all $t > 0$. On letting $n \to \infty$, we get $STx_n \to Sz$.

Now using (1),

$$h(F_{Tz,TTx_n}(\phi(ct))) \leq \psi(h(M_S(z, Tx_n))),$$

for all $t > 0$ and

$$M_S(z, Tx_n) = max\{F_{Sz,STx_n}(\phi(t)), \frac{\triangle(F_{Sz,Tz}(\phi(t)), F_{STx_n,TTx_n}(\phi(t)))}{1 + F_{Tz,TTx_n}(\phi(t))}\}.$$

On letting $n \to \infty$, we get $\lim_{n\to\infty} M_S(z, Tx_n) = 1$, and so, similar to the previous argument, we obtain

$$\lim_{n\to\infty} F_{Tz,TTx_n}(t) = 1,$$

for all $t > 0$, hence, $\lim_{n\to\infty} TTx_n = Tz$. This implies that $Sz = Tz$. So z is the coincidence point of T and S.

(Now, let h be decreasing. By the same argument, the R-weak commutativity of type-(MA_S) of S and T implies that $F_{STz,TTz}(t) \geq F_{Sz,Tz}(\frac{t}{R})$ for all $t > 0$. This means $STz = TSz = TTz = SSz$. So, by the same aforementioned argument, we obtain $Tz = TTz = STz$, i.e., Tz is the common fixed point of S and T.)

Case(c) : Let S and T be the R-weak commuting of type-(MA_T). Since S and T are weakly reciprocally continuous, hence either $\lim_{n\to\infty} STx_n = Sp$ or $\lim_{n\to\infty} TSx_n = Tp$.

Let $\lim_{n\to\infty} STx_n = Sp$. Then the R-weak commutativity of type-(MA_T) of S and T gives $F_{TSx_n,STx_{n-1}}(t) = F_{TSx_n,SSx_n}(t) \geq RF_{Tx_n,Sx_n}(t)$ for all $t>0$. Letting $n\to\infty$, we get $TSx_n \to Sp$. Similar to the previous argument in case (a) or (b), we obtain

$$\lim_{n\to\infty} F_{Tp,TTx_n}(t) = 1,$$

for all $t>0$, hence, $\lim_{n\to\infty} TTx_n = Tp$. This implies that $Sp = Tp$. So p is the coincidence point of T and S.

(Now, if h is decreasing, again by the R-weak commutativity of type-(MA_T), $F_{TSp,SSp}(t) \geq F_{Tp,Sp}(\frac{t}{R})$, which gives $TTp = STp = TSp = SSp$. So, by the same argument as in case (a) or (b), it is easy to see that $Tp = TTp = STp$, i.e., Tp is the common fixed point of S and T.)

Now, suppose that $\lim_{n\to\infty} TSx_n = Tp$. Since $T(X) \subseteq S(X)$, $Tp = Sz$ for some $z\in X$ and $\lim_{n\to\infty} TSx_n = Sz$. Since $TSx_{n+1} = TTx_n$ and $TSx_{n+1} \to Sz$, it follows that $TTx_n \to Sz$. Then the R-weak commutativity of type-(MA_T) of S and T gives $F_{TSx_n,SSx_n}(t) \geq F_{Tx_n,Sx_n}(\frac{t}{R})$ for all $t>0$. On letting $n\to\infty$, we get $SSx_n \to Sz$, and so, similar to the previous argument, we obtain

$$\lim_{n\to\infty} F_{Tz,TTx_n}(t) = 1,$$

for all $t>0$, hence, $\lim_{n\to\infty} TTx_n = Tz$. This implies that $Sz = Tz$. So z is the coincidence point of T and S.

(Now, let h be decreasing. The R-weak commutativity of type-(MA_T) of S and T implies that $F_{TSz,SSz}(t) \geq F_{Tz,Sz}(\frac{t}{R})$ for all $t>0$. This means $STz = TSz = TTz = SSz$. So, by the same argument as the one above we obtain $Tz = TTz = STz$, i.e., Tz is the common fixed point of S and T.) □

Here, we first recall the concept of weakly commuting for two mappings $T: X\times X \to X$ and $g: X\to X$ on a Menger PM space X.

Definition 15. [8] Let $(X,F,\triangle)$ be a Menger PM space and $T: X\times X\to X$ and $g: X\to X$. Then T and g are called R-weakly commuting of type-(MA_g), if there exists some real number $R\geq 0$ such that

$$F_{T(T(x,y),T(y,x)),gT(x,y)}(t) \geq F_{T(x,y),gx}(\frac{t}{R}),$$

and

$$F_{T(T(y,x),T(x,y)),gT(y,x)}(t) \geq F_{T(y,x),gy}(\frac{t}{R}),$$

for all $t>0$ and $(x,y)\in X\times X$.

Definition 16. [8] Let $(X,F,\triangle)$ be a Menger PM space and $T: X\times X\to X$ and $g: X\to X$. Then T and g are called R-weakly commuting of type-(MA_T), if there exists some real number $R\geq 0$ such that

$$F_{T(gx,gy),ggx}(t) \geq F_{T(x,y),gx}(\frac{t}{R}),$$

and

$$F_{T(gy,gx),ggy}(t) \geq F_{T(y,x),gy}(\frac{t}{R}),$$

for all $t > 0$ and $(x, y) \in X \times X$.

Reciprocal continuity and weakly reciprocal continuity are generalized for a pair of single-valued maps in the Menger PM space as follows.

Definition 17. [8] Let $(X, F, \triangle)$ be a Menger PM space and $T : X \times X \to X$ and $g : X \to X$. Then T and f are called reciprocally continuous, if $\lim_{n\to\infty} fT(x_n, y_n) = fx$, $\lim_{n\to\infty} fT(y_n, x_n) = fy$ and $\lim_{n\to\infty} T(fx_n, fy_n) = T(x, y)$, whenever $\{(x_n, y_n)\}$ is a sequence in $X \times X$ such that $\lim_{n\to\infty} T(x_n, y_n) = \lim_{n\to\infty} fx_n = x$ and $\lim_{n\to\infty} T(y_n, x_n) = \lim_{n\to\infty} fy_n = y$ for some $(x, y) \in X \times X$.

Definition 18. [8] Let $(X, F, \triangle)$ be a Menger PM space and $T : X \times X \to X$ and $g : X \to X$. Then T and f are called weakly reciprocally continuous, if $\lim_{n\to\infty} fT(x_n, y_n) = fx$ and $\lim_{n\to\infty} fT(y_n, x_n) = fy$ or $\lim_{n\to\infty} T(fx_n, fy_n) = T(x, y)$, whenever $\{(x_n, y_n)\}$ is a sequence in $X \times X$ such that $\lim_{n\to\infty} T(x_n, y_n) = \lim_{n\to\infty} fx_n = x$ and $\lim_{n\to\infty} T(y_n, x_n) = \lim_{n\to\infty} fy_n = y$ for some $(x, y) \in X \times X$.

Theorem 3. Let $(X, F, \triangle)$ be a G-complete Menger PM space with a T-norm $\triangle$ of H-type, $G : X \times X \to X$ and $f : X \to X$ be two weakly reciprocally continuous mappings satisfying (10) and (11) for some $c \in (0, 1), \phi \in \Phi, \psi \in \Psi$, and $h \in \mathcal{H}$, with $G(X \times X) \subseteq f(X)$, then G and f have a coupled coincidence point in X (if h is decreasing, G and f have a common coupled fixed point in X) if either

a. G and f are Menger compatible; or,

b. G and f are R-weakly commuting of type-(MA_f); or,

c. G and f are R-weakly commuting of type-(MA_G).

Proof. It follows from the proof of Corollary 2, that $T, S : \tilde{X} \to \tilde{X}$ are two self mappings on $\tilde{X} = X \times X$ such that a coupled coincidence point (a common coupled fixed point) of G and f is a coincidence point (common fixed point) of T and S in $X \times X$ and vice versa.

On the other hand, following an argument similar to the one used in the proof of Corollary 2, we have

$$h(\tilde{F}_{T\tilde{a},T\tilde{b}}(\phi(ct))) \leq \psi(h(M_S(\tilde{a}, \tilde{b}))),$$

for all $t > 0$ and $\tilde{a} = (x, y), \tilde{b} = (u, v) \in \tilde{X}$.

It is easy to see that T and S are weakly reciprocally continuous self-mappings of $\tilde{X}$ (and Menger compatible), if G and f are weakly reciprocally continuous (and Menger compatible).

Also, if G and f are the R-weak commuting of type-(MA_f) (or type-(MA_G)), we can prove that T and S are R-weakly commuting of type-(MA_S) (or type-(MA_T)).

Thus, from Theorem 2, we see that G and f have a coupled coincidence point (common coupled fixed point). That is, $G(p, q) = fp(= p)$ and $G(q, p) = fq(= q)$ for some $(p, q) \in X \times X$. □

Example 2. Let $X = \{2^n : n \in \mathbb{N}\} \bigcup \{0\}$ and define the mapping $F : X \times X \to \Lambda^+$ by $F_{x,y}(0) = 0$ for all $x, y \in X$, $F_{x,x}(t) = 1$ for all $x \in X$ and $t > 0$,

$$F_{x,y}(t) = \begin{cases} \frac{3}{5}, & \text{if} \quad 0 < t \leq |x-y|, \\ 1, & \text{if} \quad t > |x-y|, \end{cases}$$

for all $x, y \in X$ with $x \neq y$. It is easy to see that $(X, F, \triangle_m)$ is a complete Menger PM space.

Let $G : X \times X \to X$ and $f : X \to X$ be two mappings defined by

$$G(x, y) = 0,$$

for all $x, y \in X$ with $xy = 0$,

$$G(2, y) = 0,$$

for all $y \in X$,

$$G(x, y) = x,$$

for all $x, y \in X$ with $x \neq y$ and $x \neq 2$ and

$$f(0) = 0, \quad f(2^n) = 2^{n+1},$$

for each $n \in \mathbb{N}$. It is easy to see that $G(X \times X) = f(X) = \{2^{n+1} : n \in \mathbb{N}\} \bigcup \{0\}$ and so $f(X)$ is complete. We also see that G and f are weakly reciprocally continuous and compatible.

Now, consider the self-mappings ϕ and ψ on $[0, \infty)$ defined by $\psi(t) = \phi(t) = t$ for all $t \in [0, \infty)$, and let $h : (0, 1] \to [0, \infty)$ be an arbitrary strictly decreasing bijection between $(0, 1]$ and $[0, \infty)$ such that h and h^{-1} are continuous. In this context, the contractivity conditions (10) and (11) are equivalent to

$$\begin{aligned} & h(F_{G(x,y),G(u,v)}(\phi(ct))) \leq \psi(h(M_f^*((x,y),(u,v)))) \\ \Leftrightarrow\ & h(F_{G(x,y),G(u,v)}(ct)) \leq h(M_f^*((x,y),(u,v))) \\ \Leftrightarrow\ & F_{G(x,y),G(u,v)}(ct) \geq M_f^*((x,y),(u,v)) \\ \Leftrightarrow\ & F_{G(x,y),G(u,v)}(ct) \geq min\{F_{fx,fu}(t), F_{fy,fv}(t)\}. \end{aligned} \tag{12}$$

If $c = \frac{1}{2}$, for all $x, y, u, v \in X$, if $xy = 0$ and $uv = 0$, then G and f satisfy (8). For all $x, y, u, v \in X$ with $xy \neq 0$ or $uv \neq 0$ and $t > 0$, if $\frac{1}{2} > |G(x, y) - G(u, v)|$, then we have

$$F_{G(x,y),G(u,v)}(\frac{t}{2}) = 1 \geq min\{F_{fx,fu}(t), F_{fy,fv}(t)\}.$$

Next, assume that $\frac{1}{2} \leq |G(x, y) - G(u, v)|$. We show the condition (8) by the following cases:

I. $xy = 0, u = 2^n, v = 2^m$. For all $t > 0$, $\frac{t}{2} < |G(x, y) - G(u, v)| = 2^n$ implies that $t < 2^{n+1} = f(u)$ and so

$$F_{G(x,y),G(u,v)}(\frac{t}{2}) = \frac{3}{5} = min\{F_{fx,fu}(t), F_{fy,fv}(t)\}.$$

II. $xy \neq 0$ and $uv \neq 0$. Let $x = 2^k, y = 2^l, u = 2^n$ and $v = 2^m$ for every $k,l,n,m \in \mathbb{N}$. For all $t > 0$, $\frac{t}{2} < |G(x,y) - G(u,v)| = |2^k - 2^n|$ implies that $t < |2^{k+1} - 2^{n+1}| = |f(x) - f(u)|$ and so

$$F_{G(x,y),G(u,v)}(\frac{t}{2}) = \frac{3}{5} = min\{F_{fx,fu}(t), F_{fy,fv}(t)\}.$$

By the cases above, (10) holds for all $x,y,u,v \in X$ and all $t > 0$. Therefore, by Theorem (3), G and f have a common coupled fixed point in X. That is, there exist $x^*, y^* \in X$ such that $G(x^*,y^*) = f(x^*) = x^*$ and $G(y^*,x^*) = f(y^*) = y^*$. In fact, $x^* = y^* = 0$.

4 Conclusions

In this paper, we established the existence of a fixed point and its uniqueness for a self mapping under more general contractivity conditions replacing the function $t \to \frac{1}{t} - 1$ by an appropriate function h in a Menger probabilistic metric space. Also, we investigated the existence of a common fixed point and a coincidence point for mappings satisfying generalized (ϕ,ψ)-weak contraction condition in the same space. Moreover, our results of (ϕ,ψ)-contractions in complete Menger probabilistic metric spaces were used in differential, integral, and functional equations for optimization problems [17] and generalized some corollaries of [4, 7, 8].

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